A MAPAEKF-SLAM ALGORITHM WITH RECURSIVE MEAN AND COVARIANCE OF PROCESS AND MEASUREMENT NOISE STATISTIC

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Abstract -- *The most popular filtering method used for solving a Simultaneous Localization and Mapping is the Extended Kalman Filter. Essentially, it requires prior stochastic knowledge both the process and measurement noise statistic. In order to avoid this requirement, these noise statistics have been defined at the beginning and kept to be fixed for the whole process. Indeed, it will satisfy the desired robustness in the case of simulation. Oppositely, due to the continuous uncertainty affected by the dynamic system under time integration, this manner is strongly not recommended. The reason is, improperly defined noise will not only degrade the filter performance but also might lead the filter to divergence condition. For this reason, there has been a strong manner well-termed as an adaptive-based strategy that commonly used to equip the classical filter for having an ability to approximate the noise statistic. Of course, by knowing the closely responsive noise statistic, the robustness and accuracy of an EKF can increase. However, most of the existed Adaptive-EKF only considered that the process and measurement noise statistic are characteristically zero-mean and responsive covariances. Accordingly, the robustness of EKF can still be enhanced. This paper presents a proposed method named as a MAPAEKF-SLAM algorithm used for solving the SLAM problem of a mobile robot, Turtlebot2. Sequentially, a classical EKF was estimated using Maximum a Posteriori. However, due to the existence of unobserved value, EKF was also smoothed one time based on the fixed-interval smoothing method. This smoothing step aims to keep-up the derivation process under MAP creation. Realistically, this proposed method was simulated and compared to the conventional one. Finally, it has been showing better accuracy in terms of Root Mean Square Error (RMSE) of both Estimated Map Coordinate (EMC) and Estimated Path Coordinate (EPC).*

Keywords: Simultaneous Localization and Mapping; Adaptive Extended Kalman Filter; Maximum a Posteriori; Fixed-Interval Smoothing Method; Root Mean Square Error

INTRODUCTION

In a complex environment, a map is useful for autonomous robot navigation. However, the robot has no knowledge about the environment at the beginning. Therefore, to appropriately perform the navigation tasks, the autonomous robot should have the ability to build its map and simultaneously locate its current position. It is well-known as simultaneous localization and mapping (SLAM), which was first introduced in 1988 [1][2]. Recently, the SLAM-based mobile robot navigation has intensively received much

attention because of some challenging factors such as continuous uncertainty, system complexity, inaccurate system model, limited prior information, noise statistics of the process and measurement, computational cost and filter divergence which are required to be addressed [3, 4, 5]. An effort familiarly used to obtain an effective solution for the SLAM problem is proposed the probability-based that has been effectively and commonly used by others researchers such as Extended Kalman Filter [6, 7, 8].

Extended Kalman Filter (EKF) has become a popular choice to solve SLAM problems. Weingarten and Siegwart [9] used EKF for solving the full SLAM problem of AUV in especially estimating the local path traveled by the robot while forming the scan as well as its uncertainty and keeping the scans pose. Weingarten and Siegwart utilized an EKF as the SLAM-algorithm to track the robot when it moves and to incrementally update the Symmetric and Perturbation model (SPmodel) of the 3D reconstructed stochastic map [9]. Lemaire, Lacroix, and Solà adopted EKF to solve the bearing only 3D SLAM by estimating the parameters used for landmark initialization process such as the visual point features tracked in the sequence of the acquired images [10]. Moreover, regarding survey consequences, Dissanayake, Newman, Clark, Durrant-Whyte, and Csorba have proven the SLAM problems with the performance of EKF by developing zero uncertainty of estimated map and absolute accuracy of the map [11]. Huang, Mourikis, and Roumeliotis have demonstrated the improvement by analyzing the issue of filter consistency of extended Kalman filter based SLAM, from an observability perspective [12]. By examining the observability properties of the nonlinear SLAM system model with the linearized error-state model employed in the EKF.

The observable subspace of the standard EKF is constantly of higher dimension than the observable subspace of the underlying nonlinear system. Ahmad and Namerikawa declared that EKF based mobile robot localization with intermittent measurements is examined by analyzing the measurement innovation characteristics [13]. The uncertainties bound the estimation by analyzing the measurement innovation to preserve good estimations, although sometimes measurement data are missing. They also have proposed the theoretical analysis of the EKF to show the situations during the problem that occurred. Besides, the Jacobian transformation is considered one of the main factors to affect the estimation performance. In addition, initial state covariance, process, and measurement noises must be less to execute better estimation results. Wang, Wu, Zhou, and He have used State Transformation Extended Kalman Filter (ST-EKF) mechanization method to resolve the inconsistency problem of EKF, which is improving the propagation rates of the system matrix and the error covariance matrix [14]. However, these proposed methods cannot estimate the noise statistics. Besides that, an EKF has complications of a slow convergence rate, low accuracy, and poor numerical stability as mentioned by Gadsdenin 2011 [3, 5, 15]. For this reason, the adaptive-based approaches have been popularly attempted and utilized nowadays.

Akhlaghi, Zhou, and Huan have proposed the AEKF, which can approximate covariance matrices of the process and measurement noise statistics. It was conducted by referring to innovation and residual for improving the dynamic state estimation accuracy of EKF [16]. Jetto, Longhi, and Venturini have proposed an adaptive EKF for optimizing the linearization between KF and EKF [17]. They believed that it could be conducted by adjusting the input and measurement noise covariance matrices. Chatterjee and Matsuno have proposed the AEKF based on neuro-fuzzy to estimate the elements of covariance matrices [18]. It aims to reduce the mismatch between the theoretical and actual covariance of the innovation sequences. Moreover, Yuzhen, Quande, and Benfa have proposed AEKF by using the Sage-Husa time-varying noise estimator and Taylor series of sampling time in AEKF to estimate observation noise in real-time [19]. It aims to overcome the linearization error and enhance environmental adaptability.

Similarly, to provide proper filtering method-based solution for the SLAM problem, this paper also presents a proposed method named as A MAPAEKF based SLAM algorithm (MAPAEKF-SLAM Algorithm). Initially, the classical EKF was estimated by utilizing the creation of Maximum a Posterior (MAP) aims to approximate the posterior values of both the process and measurement noise statistic with their corresponding covariances. A smoothed EKF estimate value continuously tuned the suboptimal estimated values. It is obtained based on a fixed-point smoothing method as utilized by Caballero, Hermoso, Jiménez, and Linares in 2003 [20] as well as by Gao, Li, Zhou, and Li in 2015 [21]. Next, the estimated values were mathematically derived to obtain the time-varying noise statistic. However, due to the complexity of deriving these parameters, some certain approach was also involved.

Moreover, the proposed method is approached to address the SLAM problem of the wheeled mobile robot. Then it was utilized and validated under the robot navigation performance. Adaptively, the role of Maximum a Posterior was used to estimate the unknown parameter of a classical EKF for both the process and measurement noise statistic with their corresponding covariances. Henceforth, it

is termed as MAPAEKF-SLAM. It was simulated and compared with the classical algorithm in terms of RMSE of estimating both path and map namely, EPC and EMC, respectively. The comparative simulated result has been realistically proving that the proposed method has good and better accuracy, stability and effectiveness.

The rest parts of this paper are organized as follows. Section II contains a classical EKF. Section III presents the adaptive EKF with process derivation of the first solution given by MAP, improved/modified EKF, weight exponent method. Section IV presents the MAPAEKF-SLAM algorithm which is expanded with the discussion of the motion model, direct point-based observation, and inverse point-based observation. Section V presents some comparative results and discussion. Section VI presents the conclusion.

METHOD
Classical Extended Kalman Filter (EKF)

Through this paper, EKF is considered as the main filtering method used to estimate the path traveled by the robot when it moves. Besides that, it is utilized to locate all the features on the environment at the same time. Since the localization of the robot is only for recording the current pose of the robot and single-location of the feature/landmark, it can declare that this work is focused on solving the online SLAM problem.

Considering the nonlinear system has a model as shown below

$$\begin{cases} x_k = f(x_{k-1}, u_k) + \omega_{k-1} \\ z_k = h(x_k) + \nu_k \end{cases} \tag{1}$$

where k refers to discrete-time index, $x \in \mathbb{R}^n$ is state vector, u is control vector, $z \in \mathbb{R}^m$ is measurement vector, $\omega \in \mathbb{R}^n$ and $\nu \in \mathbb{R}^m$ are small adaptive process and measurement noise, respectively. While, $f(.)$ and $h(.)$ are the nonlinear function and measurement model, respectively. The characteristic of this dynamic model (1) is described as follows.

$$\begin{cases} E[\omega_k] = q_k, Cov[\omega_k, \omega_j] = Q_k \delta_{kj} \\ E[\nu_k] = r_k, Cov[\nu_k, \nu_j] = R_k \delta_{kj} \\ E[\omega_k, \nu_j] = 0 \end{cases} \tag{2}$$

where δ is Kronecker delta function. Whereas, $E[.]$ and $Cov[,]$ are mean and covariance term, respectively. (2) illustrates that the mean values of the process and measurement noise are nonzero mean but instead q and r, respectively. At this point, the classical of EKF can be presented as follows.

$$\hat{x}_{k|k-1} = f(\hat{x}_{k-1|k-1}, u_k) + q_{k-1} \tag{3}$$
$$P_{k|k-1} = F_{k-1} P_{k-1|k-1} F_{k-1}^T + Q_{k-1} \tag{4}$$
$$\hat{z}_{k|k-1} = h(\hat{x}_{k|k-1}) + r_k \tag{5}$$
$$e_{z,k|k-1} = z_k - \hat{z}_{k|k-1} \tag{6}$$
$$S_k = H P_{k|k-1} H^T + R_k \tag{7}$$
$$K_k = P_{k|k-1} H^T S_k^{-1} \tag{8}$$
$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k e_{z,k|k-1} \tag{9}$$
$$P_{k|k} = (I - K_k H) P_{k|k-1} \tag{10}$$

where Q and R refer to the process and measurement covariance matrix, respectively. While F refers to a Jacobian matrix of the transition function $f(.)$ with respect to prior state vector $\hat{x}_{k-1|k-1}$ and H refers to a Jacobian matrix of the measurement function $h(.)$ concerning predicted state vector $\hat{x}_{k|k-1}$.

MAP based Adaptive EKF (MAPAEKF)

Classical EKF requires the known noise statistic and an accurate system model. Unfortunately, the noise is partially known, and the model is might change because of continued uncertainty in time integration. Thus, an improvement of EKF is required as presented in this paper. An adaptive filter strategy approached to the classical EKF aims to provide the ability to approximate the noise statistic under time changes. It can be described as follows

- **Suboptimal MAP of Adaptive EKF**

Initially, a classical EKF was estimated by using MAP creation, as mentioned on [21, 22, 23, 24, 25]. Assuming that, the unknown parameters are the process q_k and measurement r_k noise statistics with their covariance Q_k and R_k, respectively. Moreover, since Q_k and R_k are considered be a positive definite symmetric matrix, then the estimated value of q_k, r_k, Q_k, and R_k can be obtained by calculating the maximum value of the following objective function L as described below

$$L = p(X_k, q, r, Q, R | Z_k) \tag{11}$$

where $X_k = [x_1, , x_k]$ and $Z_k = [z_1, , z_k]$. Referring to Bayes rule, (11) can be reformulated as follows

$$\frac{p[X_k, q, r, Q_k, R_k, Z_k]}{p[Z_k]} \tag{12}$$

Since $p[Z_k]$ plays no role in optimization, we have

$$L = p[Z_k | X_k, q, r, Q, R] p[X_k | q, Q, r, R] p[q, r, R, Q] \tag{13}$$

Assuming that $p[q, r, Q, R]$ can be obtained from the prior information means it can be regarded as being a constant. Then the posteriori distribution $p[X_k, q, r, Q, R | Z_k]$ can be calculated by

multiplying $p[Z_k|X_k, q, r, Q, R]$ with $p[X_k, q, r, Q, R|Z_k]$ as derived below

Note that q and r refer to q_{k-1} and r_k, respectively. Whereas, Q and R refer to Q_{k-1} and R_k, respectively.

Since (1) is the first-order Markov Process, then $p[X_k|q, Q, r, R]$ can be reformulated as follows

$$p[X_k|q, Q, r, R] = p[x_0] \prod_{i=1}^{k} p[x_i|x_i - 1, q, Q] \qquad (14)$$

then by assuming that (14) is normally distributed then it yields

$$p[X_k|q, r, Q, R] = \frac{1}{(2\pi)^{\frac{n}{2}} |P_0|^{\frac{1}{2}}} exp\Big[-\frac{1}{2}\|x_0 - \hat{x}_0\|_{P_0^{-1}}^2 \Big]$$

$$\prod_{i=1}^{k} \frac{1}{(2\pi)^{\frac{n}{2}} |Q|^{\frac{1}{2}}} exp\Big[-\frac{1}{2}\|x_i - f(x_i) - q\|_{Q^{-1}}^2 \Big]$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}} |P_0|^{\frac{1}{2}}} exp\Big[-\frac{1}{2}\|x_0 - \hat{x}_0\|_{P_0^{-1}}^2 \Big]$$

$$\frac{1}{(2\pi)^{\frac{nk}{2}} |Q|^{\frac{k}{2}}} exp\Big[-\frac{1}{2}\sum_{i=1}^{k}\|x_i - f(x_i) - q\|_{Q^{-1}}^2 \Big]$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}} (2\pi)^{\frac{nk}{2}}} |P_0|^{-\frac{1}{k}} |Q|^{-\frac{k}{2}}$$

$$exp\Big\{ -\frac{1}{2}\Big[\|x_0 - \hat{x}_0\|_{P_0^{-1}}^2 + \sum_{i=1}^{k}\|x_i - f(x_i) - q\|_{Q^{-1}}^2 \Big]\Big\}$$

$$= \frac{1}{(2\pi)^{\frac{nk+n}{2}}} |P_0|^{-\frac{1}{2}} |Q|^{-\frac{k}{2}}$$

$$exp\Big\{ -\frac{1}{2}\Big[\|x_0 - \hat{x}_0\|_{P_0^{-1}}^2 + \sum_{i=1}^{k}\|x_i - f(x_i) - q\|_{Q^{-1}}^2 \Big]\Big\} \qquad (15)$$

Similarly, for p[Z_k|X_k,q,r,Q,R] we have

$$p[Z_k|X_k, q, r, Q, R] = \prod_{i=1}^{k} p[z_i|x_i, r, R] \qquad (16)$$

Then assuming that (16) is Gaussian distribution then

$$p[Z_k|X_k, q, r, Q, R] =$$

$$\prod_{i=1}^{k} \frac{1}{(2\pi)^{\frac{m}{2}} |R|^{\frac{1}{2}}} exp\Big[-\frac{1}{2}\|z_i - h(x_i) - r\|_{R^{-1}}^2 \Big]$$

$$= \frac{1}{(2\pi)^{\frac{m}{2}}} |R|^{-\frac{k}{2}} exp\Big[-\frac{1}{2}\sum_{i=1}^{k}\|z_i - h(x_i) - r\|_{R^{-1}}^2 \Big] \qquad (17)$$

Then by multiplying (15), (17) and $p[q, r, Q, R]$ we have the following equation

$$p[Z_k|X_k, q, r, Q, R]p[X_k|q, Q, r, R]p[q, r, R, Q]$$

$$= \frac{1}{(2\pi)^{\frac{m}{2}}} |R|^{-\frac{k}{2}} exp\Big[-\frac{1}{2}\sum_{i=1}^{k}\|z_i - h(x_i) - r\|_{R^{-1}}^2 \Big]$$

$$\frac{1}{(2\pi)^{\frac{nk+n}{2}}} |P_0|^{-\frac{1}{2}} |Q|^{-\frac{k}{2}} exp\Big\{ -\frac{1}{2}\Big[\|x_0 - \hat{x}_0\|_{P_0^{-1}}^2 +$$

$$\sum_{i=1}^{k}\|x_i - f(x_i) - q\|_{Q^{-1}}^2 \Big]\Big\} \qquad (18)$$

Next, by supposing that

$$C = \frac{1}{(2\pi)^{\frac{mk}{2}}} \frac{1}{(2\pi)^{\frac{nk+n}{2}}} |P_0|^{-\frac{1}{2}} p[q, r, Q, R] exp\Big[\frac{1}{2}$$

$$\|x_0 - \hat{x}_0\|_{P_0^{-1}}^2 \Big] \qquad (19)$$

Then (18) can be simplified as follows

$$L = C|R|^{-\frac{k}{2}} |Q|^{-\frac{k}{2}} exp\Big\{ -\frac{1}{2}\Big[\sum_{i=1}^{k}\|z_i - h(x_i) - r\|_{R^{-1}}^2 +$$

$$\sum_{i=1}^{k}\|x_i - f(x_i) - q\|_{Q^{-1}}^2 \Big]\Big\} \qquad (20)$$

At this point, the estimated unknown parameter can be calculated by taking the logarithm of the objective function L, calculating the first derivative of logarithm L with respect to q, r, Q, and R and equating its derived to be zero. These steps can be derived as follows.

Since the logarithm of L is

$$ln(L) = ln(C) - \frac{k}{2}ln|R| - \frac{k}{2}ln|Q| -$$

$$\frac{1}{2}\Big[\sum_{i=1}^{k}\|z_i - h(x_i) - r\|_{R^{-1}}^2 + \sum_{i=1}^{k}\|x_i - f(x_i) - q\|_{Q^{-1}}^2 \Big] \qquad (21)$$

then $\hat{q}_k, \hat{r}_k, \hat{Q}_k,$ and $\hat{R}_k$ are

$$\hat{q}_k = \frac{\partial ln(L)}{\partial q} = \frac{1}{k}\sum_{i=1}^{k} x_{i|k} - f(x_{i-1|k}) \qquad (22)$$

$$\hat{r}_k = \frac{\partial ln(L)}{\partial r} = \frac{1}{k}\sum_{i=1}^{k} z_i - h(x_{i|k}) \qquad (23)$$

$$\hat{Q}_k = \frac{\partial ln(L)}{\partial Q} = \frac{1}{k}\sum_{i=1}^{k} \Big(x_{i|k} - f(x_{i-1|k}) - q_{i-1}\Big)\Big(x_{i|k} - f(x_{i-1|k}) - q_{i-1}\Big)^T \qquad (24)$$

$$\hat{R}_k = \frac{\partial ln(L)}{\partial R} = \frac{1}{k}\sum_{i=1}^{k} \Big(z_i - h(x_{i|k}) - r_i\Big)\Big(z_i - h(\hat{x}_{i|k}) - r_i\Big)^T \qquad (25)$$

The complicated multistep smoothing term $x_{i|k}$ and $x_{i-1|k}$ in (21) - (25) might cause inefficiency of the MAP estimate. Therefore, to find the conventional and efficient recursive form the simplification is needed. Note that the recursive update process only utilizes the estimated value at time k-1 and k, hence the simplification can be conducted by replacing $x_{i-1|k}$ with $\hat{x}_{i-1|i}$ in (22) and (24) and $x_{i|k}$ with $\hat{x}_{i|i}$ in (22) - (25). Therefore, the suboptimal of MAP noise estimator can be expressed as follows

$$\hat{q}_k = \frac{1}{k}\sum_{i=1}^{k}\hat{x}_{i|i} - f(\hat{x}_{i-1|i}) \tag{26}$$

$$\hat{r}_k = \frac{1}{k}\sum_{i=1}^{k} z_i - h(\hat{x}_{i|i}) \tag{27}$$

$$\hat{Q}_k = \frac{1}{k}\sum_{i=1}^{k}\Big(\hat{x}_{i|i} - f(\hat{x}_{i-1|i}) - q_{i-1}\Big)\Big(\hat{x}_{i|k} - f(\hat{x}_{i-1|i}) - q_{i-1}\Big)^T \tag{28}$$

$$\hat{R}_k = \frac{1}{k}\sum_{i=1}^{k}\Big(z_i - h(\hat{x}_{i|i}) - r_i\Big)\Big(z_i - h(\hat{x}_{i|i}) - r_i\Big)^T \tag{29}$$

As can be analyzed from the sequence equations above that the estimated value of $\hat{x}_{i-1|i}$ is not provided obviously by classical EKF. Therefore, modifying the original forms in (3)-(10) is required aims to compute the noise statistics estimator effectively

Modified and Improved EKF
     The process of modifying the EKF can be done by calculating the one-step smoothing of the EKF gain and its corresponding estimate value using the fixed point smoothing algorithm [20], [21], [26]. This process can be summarized as follows

$$\breve{x}_{k|k-1} = f(\hat{x}_{k-1|k-1}, u_k) + \hat{q}_{k-1} \tag{30}$$
$$P_{k|k-1} = F_{k-1}P_{k-1|k-1}F_{k-1}^T + \hat{Q}_{k-1} \tag{31}$$
$$\breve{z}_{k|k-1} = h(\breve{x}_{k|k-1}) + \hat{r}_k \tag{32}$$
$$\breve{e}_{z,k|k-1} = z_k - \breve{z}_{k|k-1} \tag{33}$$
$$\breve{S}_k = \breve{H}P_{k|k-1}\breve{H}^T + \hat{R}_k \tag{34}$$
$$\breve{K}_k = P_{k|k-1}\breve{H}^T\breve{S}_k^{-1} \tag{35}$$
$$\hat{x}_{k-1|k} = \breve{x}_{k-1|k-1} + \breve{K}_k\breve{e}_{z,k|k-1} \tag{36}$$

considering that the prior state $\hat{x}_{k-1|k}$ in (36) replaces the term of $\hat{x}_{k-1|k-1}$ in the normal EKF, the rest part of modified EKF are chained as follows

$$\hat{x}_{k|k-1} = f(\hat{x}_{k-1|k}, u_k) + \hat{q}_{k-1} \tag{37}$$
$$\hat{z}_{k|k-1} = h(\hat{x}_{k|k-1}) + \hat{r}_k \tag{38}$$
$$e_{z,k|k-1} = z_k - \hat{z}_{k|k-1} \tag{39}$$
$$S_k = HP_{k|k-1}H^T + \hat{R}_k \tag{40}$$

$$K_k = P_{k|k-1}H^TS_k^{-1} \tag{41}$$
$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k e_{z,k|k-1} \tag{42}$$
$$P_{k|k} = (I - K_kH)P_{k|k-1} \tag{43}$$

Now, the estimate value $\hat{x}_{i-1|i}$ and $\hat{x}_{i|i}$ can be adopted from (36) and (42), respectively.

Time-Varying Noise Statistic of EKF Based on MAP
     As can be seen above that all the suboptimal values under the mathematical derivation of MAP creation are clear. Now, both the mean of the process and measurement noise statistic and their corresponding covariance in (26)-(29), respectively, can be derived as follows. First, by substituting (42) to (26), then

$$\hat{q}_k = \frac{1}{k}\sum_{i=1}^{k}\hat{x}_{i|i} - f(\hat{x}_{i-1|i})$$
$$= \frac{1}{k}\sum_{i=1}^{k}\hat{x}_{i|i-1} + K_k e_{z,i|i-1} - f(\hat{x}_{i-1|i})$$
$$= \frac{1}{k}\sum_{i=1}^{k} f(\hat{x}_{i-1|i}) + \hat{q}_{i-1} + K_k e_{z,i|i-1} - f(\hat{x}_{i-1|i})$$
$$= \frac{1}{k}\sum_{i=1}^{k} K_k e_{z,i|i-1} + \hat{q}_{i-1} \tag{44}$$

At this point, it is obvious that the corresponding estimated covariance of $\hat{q}_k$ is

$$\hat{Q}_k = \frac{1}{k}\sum_{i=1}^{k}\big(\hat{x}_{i|i} - f(\hat{x}_{i-1|i}) - q\big)\big(\hat{x}_{i|k} - f(\hat{x}_{i-1|i}) - q\big)^T$$
$$= \frac{1}{k}\sum_{i=1}^{k}\big(K_i e_{z,i} + q - q\big)\big(K_i e_{z,i} + q - q\big)^T$$
$$= \frac{1}{k}\sum_{i=1}^{k}\big(K_i e_{z,i}\big)\big(K_i e_{z,i}\big)^T$$
$$= \frac{1}{k}\sum_{i=1}^{k} K_i e_{z,i}e_{z,i}^T K_i^T \tag{45}$$

Now, by first substituting (42) into (27), it yields

$$\hat{r}_k = \frac{1}{k}\sum_{i=1}^{k} z_i - h(\hat{x}_{i|i})$$
$$= \frac{1}{k}\sum_{i=1}^{k} z_i - h(\hat{x}_{i|i-1} + K_i e_{z,i}) \tag{46}$$

then by deriving and recalling the definition of e_i in (39), it is obtained

$$\hat{r}_k = \frac{1}{k} \sum_{i=1}^{k} z_i - h(\hat{x}_{i|i}) + HK_i e_{z,i}$$

$$= \frac{1}{k} \sum_{i=1}^{k} e_{z,i} + r_i + HK_i e_{z,i} \tag{47}$$

$$= \frac{1}{k} \sum_{i=1}^{k} (I - HK_i) e_{z,i} + r_i$$

Obviously, the estimated corresponding covariance of measurement noise is then

$$\hat{R}_k = \frac{1}{k} \sum_{i=1}^{k} \left(z_i - h(\hat{x}_{i|i}) - r_i \right) \left(z_i - h(\hat{x}_{i|i}) - r_i \right)^T$$

$$= \frac{1}{k} \sum_{i=1}^{k} \left((I - K_i H) e_{z,i} + r_i - r_i \right) \left((I - K_i H) e_{z,i} \right.$$

$$\left. r_i - r_i \right)^T$$

$$= \frac{1}{k} \sum_{i=1}^{k} (I - K_i H) e_{z,i} e_{z,i}^T (I - K_i H)^T \tag{48}$$

Note that the existence of u on the original function of $f(.)$ can be ignored at this estimation process because it is known. Besides that, it does not play any role in the optimization as well.

Finally, the recursive noise statistic and their corresponding covariance can be presented as follows

$$\hat{q}_k = \hat{q}_{k-1} + \frac{1}{k}(K_k e_{z,k}) \tag{49}$$

$$\hat{r}_k = \hat{r}_{k-1} + \frac{1}{k}\left[(I - HK_k) e_{z,k}\right] \tag{50}$$

However, to obtain the corresponding recursive covariance of (49) and (50), some derivation is required. This process can be presented as follows

Since the update covariance (43) is originally adopted from the following Joseph form

$$P_{k|k} = (I - K_k H) P_{k|k-1} (I - K_k H)^T + K_k R_k K_k^T \tag{51}$$

then the recursive of the form (45) can be alternatively reformatted as follows (see appendix A)

$$\hat{Q}_k = \frac{1}{k} \sum_{i=1}^{k} P_{i|i} + HK_i P_{i|i-1} +$$

$$P_{i|i-1} H^T K_i^T - F P_{i-1|i-1} F^T - Q_{i-1}$$

$$= \frac{k-1}{k} Q_{k-1} + \frac{1}{k} \left[P_{k|k} + HK_k P_{k|k-1} + \right. \tag{52}$$

$$\left. P_{k|k-1} K_k^T H^T - F P_{k-1|k-1} F^T \right]$$

Similarly, for the recursive covariance of the measurement noise statistic can be obtained as follows (see appendix B)

$$\hat{R}_k = \frac{1}{k} \sum_{i=1}^{k} HP_{i|i-1} H^T + R_i - HK_i e_{z,i} e_{z,i}^T -$$

$$e_{z,i} e_{z,i}^T K_i^T H^T + HK_i e_{z,i} e_{z,i}^T K_i^T H^T$$

$$= \frac{k-1}{k} R_{k-1} + \frac{1}{k} \left[HP_{k|k-1} H^T - HK_k e_{z,k} e_{z,k}^T - \right.$$

$$e_{z,k} e_{z,i}^T K_k^T H^T + HK_k e_{z,k} e_{z,k}^T K_k^T H^T \Big] \tag{53}$$

A MAP Based AEKF-SLAM Algorithm

The proposed method is applied for solving the SLAM problem of wheeled mobile robots. Henceforth it is termed as AEKF-SLAM Algorithm Based on MAP-WE. Basically, it can be solved by locating the current robot position and gathering the information of features in a certain environment. Therefore, the movement and observation method is required. Both are discussed in the following subsection.

Motion Model

$$x_{R,k} = \begin{cases} x_R^A & v_r = v_l \\ x_R^B & v_r \neq v_l \end{cases} \tag{54}$$

By supposing that, the robot is simulated and initially located in a certain planar environment. Then the kinematic configuration of the robot movement can be shown in Figure 1.
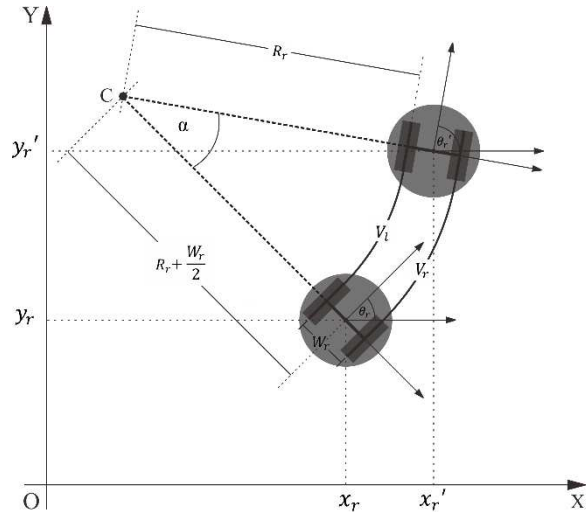

Figure 1. Kinematic Configuration

Now, considering x_r and y_r are the spatial position and θ_r is the orientation or the robot heading, then the state vector of the robot position can be expressed by $x_R = [x_r, y_r, \theta_r]^T$. The robot moves based on the odometry and differential steering system, then since the control vector contains

two-element vectors represented by $u = [u_r, u_l]^T$ for right wheel u_r and left wheel velocity u_l, then the motion model is expressed as follows

$$x_R^A = \begin{bmatrix} x_{r,k}^A \\ y_{r,k}^A \\ \theta_{r,k}^A \end{bmatrix} = \begin{bmatrix} x_{r,k-1}^{r^A} \\ y_{r,k-1}^A \\ \theta_{r,k-1}^A \end{bmatrix} + v \begin{bmatrix} cos(\theta_{r,k-1}^A) \\ sin(\theta_{r,k-1}^A) \\ 0 \end{bmatrix} \quad (55)$$

$$x_R^B = \begin{bmatrix} x_{r,k}^B \\ y_{r,k}^B \\ \theta_{r,k}^B \end{bmatrix} \quad (56)$$

$$= \begin{bmatrix} x_{r,k-1}^B \\ y_{r,k-1}^B \\ \theta_{r,k-1}^B \end{bmatrix} + \begin{bmatrix} \left(R_r + \frac{W_r}{2}\right)\left(sin(\theta_{r,k-1}^B + \alpha) - sin(\alpha \\ \left(R_r + \frac{W_r}{2}\right)\left(-cos(\theta_{r,k-1}^B + \alpha) + cos(\\ 0 \end{bmatrix}$$

where k is discrete time index, $R_r = \dfrac{v_l}{\alpha}$ represent the radius caused by the motion, W_r is the width of the robot, and $.^A$ and $.^B$ are linear and nonlinear motion model respect to the measured velocity $v = [v_r, v_l]^T$. The motion is unpredictable since the small noise follows the motion control. For this reason, the different types of motion models are introduced. Thus, the measured right-wheel v_r and left-wheel velocity v_l can be regarded as follows with the existence of the small perturbation $n = [n_r, n_l]^T$

$$v = u + n \quad (57)$$

$$\begin{cases} n_r = \zeta_1 u_r + \zeta_2(u_l - u_r) \\ n_l = \zeta_1 u_l + \zeta_2(u_l - u_r) \end{cases} \quad (58)$$

where ζ_1 is the moving factor and ζ_2 is the turning factor.

Direct Point-Based Observation

Considering that, the state vectors are composed of the robot x_R and landmark state $x_L^i = [x_{L,x}^i, x_{L,y}^i]^T$. Therefore, a full state vector is $x = [x_r, y_r, \theta_r, x_{L,x}^1, x_{L,y}^1, ..., ..., x_{L,x}^m, x_{L,y}^m]^T$ for $x_{L,x}^i$ and $x_{L,y}^i$ represent the i-th landmark position for $i = 1, 2, ..., m$.
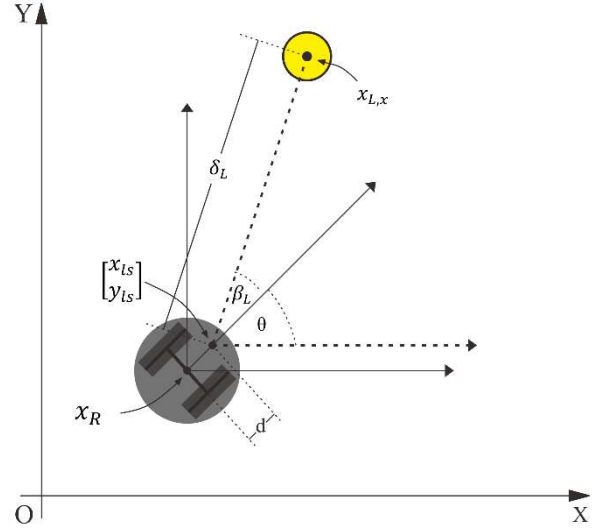

Figure 2. New Detected Landmark

Figure 2 illustrates when the robot's laser scanner detects the i-th landmark. Since $x_{ls}^R = [x_{ls}, y_{ls}]^T$ refer to the position of the laser scanner, the direct point-based observation model can be calculated as follows

$$\begin{bmatrix} x_{ls} \\ y_{ls} \end{bmatrix} = \begin{bmatrix} x_r \\ y_r \end{bmatrix} + d_{ls} \begin{bmatrix} cos(\theta_r) \\ sin(\theta_r) \end{bmatrix} \quad (59)$$

$$\begin{bmatrix} \delta_L^i \\ \beta_L^i \end{bmatrix} = \begin{bmatrix} \sqrt{(x_{L,x}^i - x_{ls})^2 + (x_{L,y}^i - y_{ls})^2} \\ arctan\left(\frac{x_{L,y}^i - y_{ls}}{x_{L,x}^i - x_{ls}}\right) \end{bmatrix} \quad (60)$$

where d_{ls} is the displacement of laser scanner and δ_L^i and β_L^i are the distance and bearing sensed by laser scanner. Next, by considering that the measurement is followed by small perturbation $r = [r_\delta, r_\beta]^T$ then we have

$$z_i = \begin{bmatrix} \delta_L^i \\ \beta_L^i \end{bmatrix} + \begin{bmatrix} r_\delta \\ r_\beta \end{bmatrix} \quad (61)$$

Now it can be noted that since (54) - (61) are gained, both state transition in (30) and (37) and measurement function in (32) and (38) are satisfied.

Inverse Point-Based Observation

A new observed landmark should be initialized and added to the state vector x. It can be conducted by using the inverse observation concept. It initiates the mapping process by utilizing the information of the current robot and landmark position. It can be written as follows

$$x_L^{new} = h^{-1}(x_{R,k}, z_i) \quad (62)$$

$$\begin{bmatrix} x_{L,x}^{new} \\ x_{L,y}^{new} \end{bmatrix} = \begin{bmatrix} x_r \\ y_r \end{bmatrix} + \delta_L^i \begin{bmatrix} cos(\theta_r)cos(\beta_L^i) - sin(\theta_r)sin(\beta_L^i) \\ cos(\theta_r)cos(\beta_L^i) + cos(\theta_r)sin(\beta_L^i) \end{bmatrix}$$
$$(63)$$

At this point, the completeness of MAPAEKF and SLAM are fully derived. Moreover, the flowchart of the whole process above can be graphically concluded as shown in Figure 3.
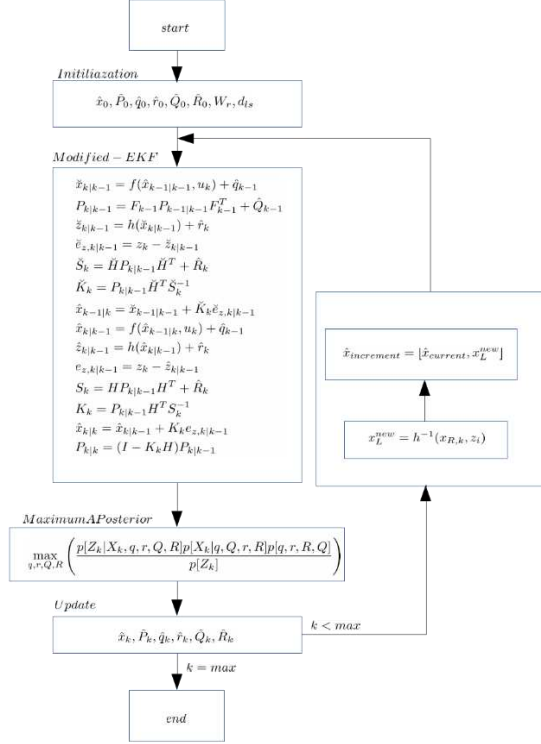

Figure 3. MAPAEKF-SLAM Algorithm

RESULTS AND DISCUSSION

In order to verify the effectiveness and accuracy, the proposed method was realistically simulated. It was compared with the classical algorithm, EKF-SLAM algorithm, in terms of RMSE of EPC and EMC. Initially, some parameters related to the robot were defined as follows

$$W_r = 33\,cm, d_{ls} = 14\,cm$$

Note that these parameters are adopted from the real robot platform that is Turtlebot2.

Furthermore, the initial state and its error covariance were also defined as follows

$$\hat{x}_0 = \begin{bmatrix} 0 \\ 0 \\ \left(\frac{35\pi}{180}\right) \end{bmatrix}, P_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

According to the initial noise statistic, different SLAM-based algorithm was performed and compared. They can be presented as follows

1st Test

The initial process and measurement noise statistic are considered as follows

$$\hat{q}_0 = \begin{bmatrix} 0.02 \\ \frac{\pi}{180} \end{bmatrix}, \hat{Q}_0 = \begin{bmatrix} (0.02)^2 & 0 \\ 0 & \left(\frac{\pi}{180}\right)^2 \end{bmatrix}$$

$$\hat{r}_0 = \begin{bmatrix} 0.15 \\ \left(\frac{3\pi}{180}\right) \end{bmatrix}, \hat{R}_0 = \begin{bmatrix} (0.15)^2 & 0 \\ 0 & \left(\frac{3\pi}{180}\right)^2 \end{bmatrix}$$

Then the result of the EKF-SLAM and MAPAEKF-SLAM algorithm can be compared based on Figure 4.
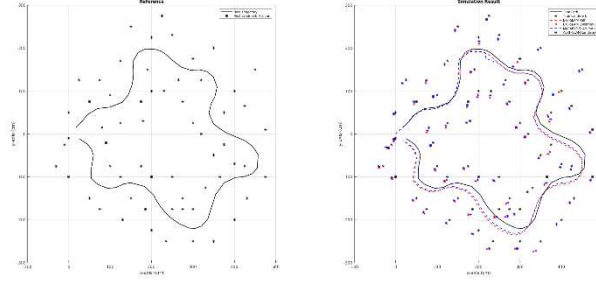

Figure 4. SLAM-Algorithm Performance (1st Test)

Figure 4 illustrates the performance of EKF and MAPAEKF. They are applied to an autonomous wheeled mobile robot for solving the SLAM problem. It depicts that the MAPAEKF-SLAM gives a better solution proven by the successful in following the reference path. For more detail, it can be analyzed by the following result.
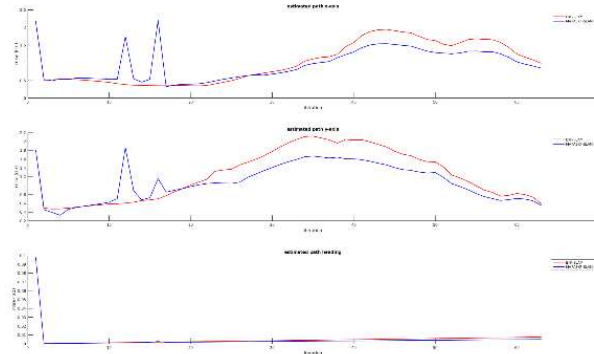

Figure 5. RMSE of Estimated Path (1st Test)

Figure 5 shows the RMSE of different performed algorithm in estimating the path. Comparing to the conventional approach, MAPAEKF-SLAM algorithm has a better accuracy pointed by the smaller RMSE in almost all benchmarks. Furthermore, an effort to provide more comparative result, the different RMSE of estimated map is also presented as follows.
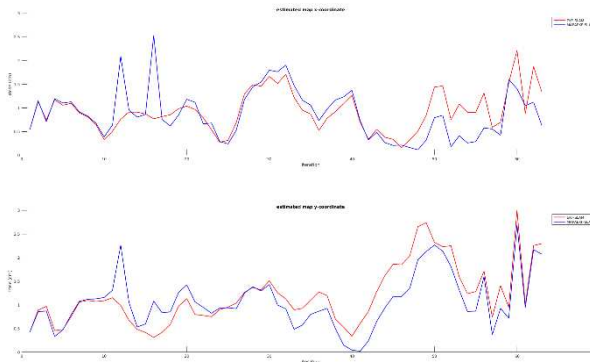
Figure 6. RMSE of Estimated MAP (1st Test)

Figure 6 shows the different quality of EKF and MAPAEKF-SLAM algorithm in estimating the location of the landmark. According to this figure, the proposed method shows better performance in the estimated map for x-coordinate and y-coordinate. To confirm this statement, Table 1 is presented

Table 1. Different SLAM Algorithm in Term of Root Mean Square Error of Estimated Path and Map

SLAM Alg.	RMSE of Estimated Path Coordinate			RMSE of Estimated Map Coordinate	
	x	y	θ	x	y
EKF	9.2529	10.825	0.1041	8.0529	10.686
AEKF	8.3448	9.1613	0.1010	8.0533	9.5750

2nd Test

Relatively, the small noise statistic both the process and measurement are unknown. For this reason, their initial predetermined value might be either small or large. Thus, the second case with the increment on those values was considered as well. It aims to validate the suitability of the proposed method for the dynamic system with the large uncertainty and unavoidable noise of the sensor and actuator. The initial process and measurement noise are considered as follows

$$\hat{q}_0 = \begin{bmatrix} 0.03 \\ \left(\frac{3\pi}{180}\right) \end{bmatrix}, \hat{Q}_0 = \begin{bmatrix} (0.03)^2 & 0 \\ 0 & \left(\frac{3\pi}{180}\right)^2 \end{bmatrix}$$

$$\hat{r}_0 = \begin{bmatrix} 0.3 \\ \left(\frac{5\pi}{180}\right) \end{bmatrix}, \hat{R}_0 = \begin{bmatrix} (0.3)^2 & 0 \\ 0 & \left(\frac{5\pi}{180}\right)^2 \end{bmatrix}$$

Like the previous experiment, the general performance of the EKF and MAPAEKF-SLAM algorithm is evaluated from the following graphical performance.
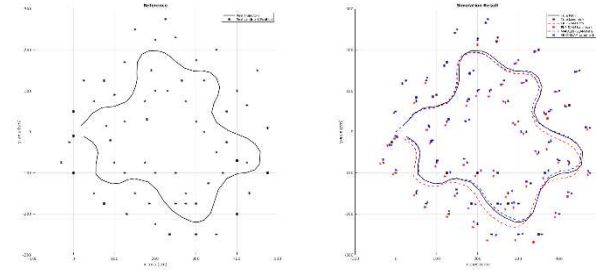

Figure 7. SLAM-Algorithm Performance (2nd Test)

Figure 7 depicts that the increment of the initial noise statistic does not affect the stability of MAPAEKF-SLAM. Therefore, it can be noted that the proposed method provides a more stable filter compared with the EKF-SLAM algorithm. For the 2nd Test, the EKF and MAPAEKF-SLAM are also performed and compared in terms of RMSE. It is depicted in Figure 8.
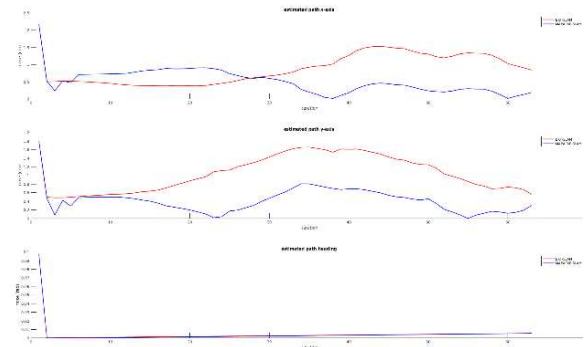
Figure 8. RMSE of Estimated Path (2nd Test)
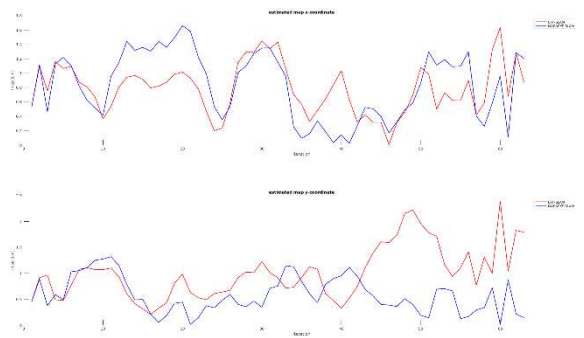
Figure 9. RMSE of Estimated MAP (2nd Test)

According to Figure 8, the MAPAEKF-SLAM algorithm shows its effectiveness in locating the current robot position. It is proven by the smaller RMSE for almost all the benchmark. Then, it can be noted that the accuracy of the MAPAEKF-SLAM algorithm is guaranteed even though there exists a noise statistic increment.

Additionally, the result of the estimated map is also presented as shown in Figure 9.

Similarly, Figure 8 shows that the MAPAEKF-SLAM has a better solution of estimating the map for x-coordinate and y-coordinate significantly. Therefore, it can be noted that the MAPAEKF-SLAM algorithm performed well in estimating the landmark. It is confirmed by Table 2.

Table 2. Different SLAM Algorithm in Term of Root Mean Square Error of Estimated Path and Map (2nd Test)

SLAM Alg.	RMSE of Estimated Path Coordinate			RMSE of Estimated Map Coordinate	
	x	y	θ	x	y
EKF	7.6909	8.9144	0.1013	6.8211	8.9208
AEKF	4.8565	3.9445	0.1009	7.5220	5.3381

Table.2 shows clearly that the MAPAEKF-SLAM algorithm provides a stable filter. Additionally, according to the result of the 1st and 2nd Test presented above, it can be noted that the MAPAEKF-SLAM has better quality in providing solutions under noise statistics increment.

CONCLUSION

This paper presents a proposed method termed as the MAPAEKF-SLAM algorithm. The contributions can be summarized as follows: estimating the noise statistic; improving the normal EKF based on one-step smoothing method; deriving the suboptimal estimate values under the creation of Maximum a Posteriori (MAP), the proposed method has been regarded to be able to solve SLAM problem even when the unknown noise statistic is large. Based on the compared result and discussion presented in the previous section, its robustness and effectiveness have been validated.

ACKNOWLEDGMENT

APPENDIX | Mathematical Derivation of The Process and Measurement Noise Statistic

Appendix A

$$P_{k|k} = (I - K_k H)P_{k|k-1}(I - K_k H)^T + K_k R_k K_k^T$$
$$= (P_{k|k-1} - K_k H P_{k|k-1})(I - K_k H)^T + K_k R_k K_k^T$$
$$= P_{k|k-1} - K_k H P_{k|k-1} - P_{k|k-1}H^T K_k^T + K_k H P_{k|k-1}H^T K_k^T + K_k R_k K_k^T \tag{A1}$$

Alternatively, it can be rewritten as followed

$$P_{k|k} = P_{k|k-1} - K_k H P_{k|k-1} - P_{k|k-1}H^T K_k^T + K_k(H P_{k|k-1}H^T + R_k)K_k^T \tag{A2}$$

Now it is clear that (A2) contains S_k. Moreover, the expectation of S_k is $[e_{x,k|k-1}e_{x,k|k-1}^T]$. Therefore, the derivation of suboptimal Q can be mathematically derived as follows

$$K_k e_{x,k|k-1}e_{x,k|k-1}^T K_k^T = P_{k|k} - K_k H P_{k|k-1} - P_{k|k-1}H^T K_k^T \tag{A3}$$

Then by substituting (31) and substituting the complete form into (45), it yields

$$\hat{Q}_k = \frac{1}{k}\sum_{i=1}^{k} P_{i|i} + HK_i P_{i|i-1} + P_{i|i-1}H^T K_i^T - FP_{i-1|i-1}F^T - Q_{i-1} \tag{A4}$$

Appendix B

$$(I - K_k H)e_{z,k|k-1}e_{z,k|k-1}^T(I - K_k H)^T$$
$$= (e_{z,k|k-1}e_{z,k|k-1}^T - HK_k e_{z,k|k-1}^T(I - K_k H)^T)$$
$$= (e_{z,k|k-1}e_{z,k|k-1}^T - HK_k e_{z,k|k-1}e_{z,k|k-1}^T - e_{z,k|k-1}e_{z,k|k-1}^T(I - K_k H)^T K_k^T H^T + HK_k e_{z,k|k-1}e_{z,k|k-1}^T(I - K_k H)^T K_k^T H^T \tag{B1}$$

Since $[e_{z,k|k-1}e_{z,k|k-1}^T]$ is partially reversed to the original form before taking the expectation of S_k, it is obvious that the following form is the alternative equation of (B1)

$$(I - K_k H)e_{z,k|k-1}e_{z,k|k-1}^T(I - K_k H)^T$$
$$= HP_{k|k-1}H + R_{k-1} - HK_k e_{z,k|k-1}e_{z,k|k-1}^T - e_{z,k|k-1}e_{z,k|k-1}^T(I - K_k H)^T K_k^T H^T + HK_k e_{z,k|k-1}e_{z,k|k-1}^T(I - K_k H)^T K_k^T H^T \tag{B2}$$

Then by substituting (B2) into (48), it yields

$$\hat{R}_k = \frac{1}{k}\sum_{i=1}^{k}(I - K_i H)e_{z,i}e_{z,i}^T(I - K_i H)^T \tag{B3}$$

$$\hat{R}_k = \frac{1}{k}\sum_{i=1}^{k} HP_{i|i-1}H + R_{i-1} - HK_i e_{z,i}e_{z,i}^T -$$
$$e_{z,i}e_{z,i}^T(I - K_iH)^T K_i^T H^T +$$
$$HK_i e_{z,i}e_{z,i}^T(I - K_iH)^T K_i^T H^T$$

(B4)

Note that the time index of $e_{z,k|k-1}$ is the representation of $e_{z,i}$ out of the summation operation. It can be simply represented as $e_{z,k}$ as well.

REFERENCES

[1] H. Durrant-Whyte and T. Bailey, "Simultaneous localization and mapping: part I," *IEEE Robot. Autom. Mag.*, vol. 13, no. 2, pp. 99–110, June 2006. DOI: 10.1109/MRA.2006.1638022

[2] T. Bailey and H. Durrant-Whyte, "Simultaneous localization and mapping (SLAM): Part II," *IEEE Robot. Autom. Mag.*, vol. 13, no. 3, pp. 108–117, September 2006. DOI: 10.1109/MRA.2006.1678144

[3] S. A. Gadsden, "Smooth Variable Structure Filtering: Theory and Applications," *Thesis,* McMaster University, August 2011.

[4] S. Habibi, "The smooth variable structure filter," *Proc. IEEE*, vol. 95, no. 5, pp. 1026–1059, May 2007. DOI: 10.1109/JPROC.2007.893255

[5] S. A. Gadsden, M. Al-Shabi, and S. R. Habibi, "A Review of Smooth Variable Structure Filters: Recent Advances Theory and and Application," *ASME International Mechanical Engineering Congress and Exposition,* 8 pages, March 2016. DOI: 10.1115/IMECE2015-50966

[6] S. Thrun, W. Burgard, and D. Fox, "Rob550: Probabilistic Robotics," pp. 1999–2000, 1999.

[7] J.-A. Fernández-Madrigal, *Simultaneous Localization and Mapping for Mobile Robots: Introduction and Methods: Introduction and Methods.* IGI Global, 2012. DOI: 10.4018/978-1-4666-2104-6

[8] S. Thrun, "Simultaneous localization and mapping," in *Robotics and cognitive approaches to spatial mapping*, Springer, 2007, pp. 13–41. DOI: 10.1007/978-3-540-75388-9_3

[9] J. Weingarten and R. Siegwart, "EKF-based 3D SLAM for structured environment reconstruction," *2005 IEEE/RSJ Int. Conf. Intell. Robot. Syst. IROS*, Edmonton, Alta, 2005, pp. 2089–2094. DOI: 10.1109/IROS.2005.154285

[10] T. Lemaire, S. Lacroix, and J. Solà, "A practical 3D bearing-only SLAM algorithm," *2005 IEEE/RSJ Int. Conf. Intell. Robot. Syst. IROS,* Edmonton, Alta, 2005, pp. 2757–2762. DOI: 10.1109/IROS.2005.1545393

[11] M. W. M. G. Dissanayake, P. Newman, S. Clark, H. F. Durrant-Whyte, and M. Csorba, "A solution to the simultaneous localization and map building (SLAM) problem," *IEEE Trans. Robot. Autom.*, vol. 17, no. 3, pp. 229–241, June 2001. DOI: 10.1109/70.938381

[12] G. P. Huang, A. I. Mourikis, and S. I. Roumeliotis, "Analysis and improvement of the consistency of extended Kalman filter based SLAM," in *2008 IEEE International Conference on Robotics and Automation*, Pasadena, CA, 2008, pp. 473–479. DOI: 10.1109/ROBOT.2008.4543252

[13] H. Ahmad and T. Namerikawa, "Extended Kalman filter-based mobile robot localization with intermittent measurements," *Syst. Sci. Control Eng.*, vol. 1, no. 1, pp. 113–126, December 2013. DOI: 10.1080/21642583.2013.864249

[14] M. Wang, W. Wu, P. Zhou, and X. He, "State transformation extended Kalman filter for GPS/SINS tightly coupled integration," *GPS Solut.*, vol. 22, no. 4, p. 112, August 2018. DOI: 10.1007/s10291-018-0773-3

[15] S. A. Gadsden, S. R.Habibi, D. Dunne, and T. Kirubarajan, "Combined Particle and Smooth Variable Structure Filtering for Nonlinear Estimation Problems [C]," *14th Int. Conf. Inf. Fusion,* Chicago, Illinois, USA, 2011. pp. 1552-1558.

[16] S. Akhlaghi, N. Zhou, and Z. Huang, "Adaptive adjustment of noise covariance in Kalman filter for dynamic state estimation," in *Power & Energy Society General Meeting, 2017 IEEE*, Chicago, IL, 2017, pp. 1–5. DOI: 10.1109/PESGM.2017.8273755

[17] L. Jetto, S. Longhi, and G. Venturini, "Development and experimental validation of an adaptive extended Kalman filter for the localization of mobile robots," *IEEE Trans. Robot. Autom.*, vol. 15, no. 2, pp. 219–229, April 1999. DOI: 10.1109/70.760343

[18] A. Chatterjee and F. Matsuno, "A neuro-fuzzy assisted extended Kalman filter-based approach for simultaneous localization and mapping (SLAM) problems," *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 5, pp. 984–997, October 2007. DOI: 10.1109/TFUZZ.2007.894972

[19] P. Yuzhen, Y. Quande, and Z. Benfa, "The application of adaptive extended Kalman filter in mobile robot localization," in *Control and Decision Conference (CCDC),* Yinchuan, China, 2016, pp. 5337–5342. DOI: 10.1109/CCDC.2016.7531952

[20] R. Caballero and A. Hermoso, "Filtering and Fixed - Point Smoothing from an Innovation Approach in Systems with Uncertainty," *Extracta Mathematicae,* vol. 18, no. 1, pp. 119–128, 2003.

[21] W. Gao, J. Li, G. Zhou, and Q. Li, "Adaptive Kalman filtering with recursive noise estimator for integrated SINS/DVL systems," *J. Navig.*, vol. 68, no. 1, pp. 142–161, January 2015. DOI: 10.1017/S0373463314000484

[22] J. He, Q. Zhang, Q. Hu, and G. Sun, "A Hybrid Adaptive Unscented Kalman Filter Algorithm," *International Journal of Engineering Modelling,* vol. 31, no. 3, pp. 51-65, January 2018. DOI: 10.31534/engmod.2018.3.ri.04d

[23] A. Logothetis and V. Krishnamurthy, "Expectation maximization algorithms for MAP estimation of jump Markov linear systems," *IEEE Trans. Signal Process.*, vol. 47, no. 8, pp. 2139–2156, August 1999. DOI: 10.1109/78.774753

[24] X.-X. WANG, M. SUN, C. YAN, J.-C. DING, and L. ZHAO, "Adaptive UKF Filtering Algorithm Based on Maximum a Posterior Estimation and Exponential Weighting," *Acta Autom. Sin.*, vol. 36, no. 7, pp. 1007–1019, July 2010. DOI: 10.3724/SP.j.1004.2010.01007

[25] Y. Tian, H. Suwoyo, W. Wang, D. Mbemba, and L. Li, "An AEKF-SLAM Algorithm with Recursive Noise Statistic Based on MLE and EM," *Journal of Intelligent & Robotic System,* vol. 97, pp. 339-355, 2020. DOI: 10.1007/s10846-019-01044-8

[26] F. Gustafsson and F. Gustafsson, *Adaptive filtering and change detection*, John Willey & Sons, Ltd., UK., 2000.