Construction of Cone 2-Norm Associated with S-Cone Inner Product

Sadjidon, Mahmud Yunus, Sunarsini, and Lukman Hanafi

***Abstract*—This paper is devoted to discussing an inner product in cone normed spaces and constructing S-cone inner products to define S-cone inner product spaces, especially in ℓ_2-space. Moreover, we also construct cone 2-norm spaces associated with S-cone inner product spaces.**

***Index Terms*—Cone Normed Spaces, S-cone inner product spaces, Cone 2-Norm Spaces**

I. Introduction

THE study of 2-norm space continues to grow and learn; among others, the study of 2-norms by associating its dual space that has been studied in [1][2][3], especially for the ℓ_2-space and the inner product space, and also in [4] by studying the cone normed space. Therefore, taking into account in [1][2] is developed a study of the cone 2-norm and some of its properties described in [5][6]. Therefore, concerning inner product space and in [4][5] it has been developed and studied about the S-cone inner product space, then they obtained the construction and definition of S-cone inner product spaces, particularly for ℓ_2-space. They also describe its properties, and construct its cone 2-normed such that is obtained a definition of cone 2-normed associated with a S-cone inner product.

To construct the S-cone inner product space, particularly for ℓ_2-space, we need and use the following definitions and notations.

Definition 1. [1] *Let X be a real vector space. A norm on X is a function $\|\cdot\| : X \to \mathbb{R}$ satisfying:*

(N1) $\|x\| \geq 0$ *for every $x \in X$ and $\|x\| = 0$ if and only if $x = 0$;*
(N2) $\|\alpha x\| = |\alpha| \, \|x\|$ *for every $x \in X$ and $\alpha \in \mathbb{R}$;*
(N3) $\|x + y\| = \|x\| + \|y\|$.

A vector space X equipped with a norm $\|\cdot\|$, written as $(X, \|\cdot\|)$, is called normed space.

Definition 2. [1] *Let X be a real vector space. An inner product on X is a function $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{R}$ that satisfies:*

(I1) $\langle x, x\rangle \geq 0$ *for every $x \in X$; and $\langle x, x\rangle = 0$ if and only if $x = 0$;*
(I2) $\langle x, y\rangle = \langle y, x\rangle$;
(I3) $\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$ *for every $x, y \in X$ and $\alpha \in \mathbb{R}$;*
(I4) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ *for every $x, y, z \in X$;*

A vector space X equipped with an inner product $\langle\cdot,]cdot\rangle$, also written as $(X, \langle\cdot,\cdot\rangle)$, is called inner product space.

Definition 3. [4] *Let P be a subset of a Banach space E with zero element θ, then P is called cone if:*

(i) *P is a closed non empty set, and $P \neq \{\theta\}$;*
(ii) *If a and b are positive real numbers, then $ax + by \in P$ for every $x, y \in P$;*
(iii) $P \cap (-P) = \{\theta\}$.

Additionally, a cone P has a relation $\preccurlyeq$ and $x \preccurlyeq y$ if and only if $y - x \in P$ and $x \prec y$ if and only if $x \prec y$ and $x \neq y$, while $x \ll y$ means $y - x \in \text{int}(P)$ (interior of P). Furthermore, we assume that E is the Banach space and P is a cone in E.

Definition 4. [4] *A cone normed space is an ordered pair $(X, \|\cdot\|_c)$ where X is a linear space over $\mathbb{R}$ and $\|\cdot\|_c : X \to (E, P, \|\cdot\|)$ is a function satisfying*

(C1) $\|x\|_c \succcurlyeq \theta$ *for every $x \in X$;*
(C2) $\|x\|_c = \theta$ *if and only if $x = 0$;*
(C3) $\|\alpha x\|_c = |\alpha| \, \|x\|_c$ *for every $x \in X$ and $\alpha \in \mathbb{R}$;*
(C4) $\|x + y\|_c \preccurlyeq \|x\|_c + \|y\|_c$ *for every $x, y \in X$.*

Definition 5. [1] *Let x be a d-dimensional real vektor space, where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ satisfying*

(N1) $\|x, y\| \geq 0$ *for ef=very $x, y \in X$; and $\|x, y\| = 0$ if and only if x and y are linearly dependent;*
(N2) $\|x, y\| = \|y, x\|$ *for every $x, y \in X$;*
(N3) $\|x, \alpha y\| = |\alpha| \, \|x, y\|$ *for every $x, y \in X$ and $\alpha \in \mathbb{R}$;*
(N4) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ *for every $x, y, z \in X$.*

A vector space X equipped with a 2-norm, also written as $(X, \|\cdot,\cdot\|)$, is called 2-norm space.

For historical issues regarding inner product spaces and 2-normed spaces, we refer to the existing references; e.g. [7], [1], [8], in which defined a standard norm:

$$\begin{aligned}\|x, y\|^2 &= \begin{vmatrix}\langle x, x\rangle & \langle x, y\rangle \\ \langle y, x\rangle & \langle y, y\rangle\end{vmatrix} \\ &= \langle x, x\rangle \langle y, y\rangle - \langle x, y\rangle^2 \\ &= \|x\|^2 \|y\|^2 - \langle x, y\rangle^2 .\end{aligned}$$

As in [2], we define the 2-norm by associating its dual space with $\langle x, z\rangle$:

$$\|x, y\| = \sup \left\{ \begin{vmatrix}\langle x, y\rangle & \langle y, z\rangle \\ \langle x, w\rangle & \langle y, w\rangle\end{vmatrix} : z, w \in \ell^2, \|z\|, \|w\| \leq 1 \right\}.$$

Geometrically, the 2-norm is the area spanned by two vectors.

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Definition 6. [5] *Let X be a 2-normed space, and $(E, \|\cdot\|)$ be a Banach space, and $P \subset E$ be a cone, then cone 2-norm on X is a function $\|\cdot,\cdot\|_C : X \times X \to (E, P, \|\cdot\|)$ satisfying the following properties:*

(CN1) $\|x, y\|_C \succcurlyeq \theta$ *for every* $x, y \in X$*; and* $\|x, y\|_C = \theta$ *if and only if* x *and* y *are linearly dependent;*
(CN2) $\|x, y\|_C = \|y, x\|_C$ *for every* $x, y \in X$*;*
(CN3) $\|\alpha x, y\|_C = |\alpha| \|x, y\|_C$ *for every* $x, y \in X$ *and* $\alpha \in \mathbb{R}$*;*
(CN4) $\|x, y+z\|_C \preccurlyeq \|x, y\|_C + \|x, z\|_C$ *for every* $x, y, z \in X$.

A 2-normed space X equipped with cone 2-norm, written as $(X, \|\cdot,\cdot\|_C)$, is called cone 2-normed space.

II. Results and Discussion

It is straightforward to verify that if $P \subset \mathbb{R}^n$ for non-negative $\mathbb{R}$, then P is a cone. As stated in [9], a function $\|\cdot\|_C : \ell_2 \to (\mathbb{R}^n, P, \|\cdot\|)$ defined by $\|x\|_C = \sum_{k=1}^{n} e_k \|x\|_{\ell_2}$ is a cone normed space. Multiplication on a cone norm is defined as follows:

$$\|x\|_C \|x\|_C = (\|x\|_C)^2 = \sum_{k=1}^{n} e_k \left(\|x\|_{\ell_2}\right)^2.$$

Let P be a subset of Banach space E and P is a cone, then we define $\mathcal{P} = P \cup (-P)$, and $\mathcal{P}$ is called S-cone. Thus, from the description of the inner product space and the meaning of S-cone, we can construct and define a S-cone inner product space as in the following definition.

Definition 7. *A S-cone inner product space is an order pair $(X, \langle\cdot,\cdot\rangle_C)$ where X a linear space over$\mathbb{R}$ with $\mathcal{P}$ is a S-cone inner product space and $\langle\cdot,\cdot\rangle_C : X \times X \to (E, \mathcal{P}, \|\cdot\|)$ is a function satisfying:*

(IC1) $\langle x, x\rangle_C \succcurlyeq \theta$ *for every* $x \in X$*; and* $\langle x, x\rangle_C = \theta$ *if and only if* $x = 0$*;*
(IC2) $\langle x, y\rangle_C = \overline{\langle y, x\rangle}_C$ *for every* $x, y \in X$*;*
(IC3) $\langle \alpha x, y\rangle_C = \alpha \langle x, y\rangle_C$ *for every* $x, y \in X$ *and* $\alpha \in \mathbb{R}$*;*
(IC4) $\langle x+y, z\rangle_C \preccurlyeq \langle x, z\rangle_C + \langle y, z\rangle_C$ *for every* $x, y, z \in X$.

If X is a real vector space, then $\overline{\langle y, x\rangle}_C = \langle y, x\rangle_C = \langle x, y\rangle_C$.

It is easy to show that ℓ_2-space with the standard inner product is a Banach space, and its S-cone inner product is given as follows.

Theorem 1. *Let $(\ell_2, \langle\cdot,\cdot\rangle)$ be an inner product space with $\mathcal{P}$ is a S-cone, and we define a function*

$$\begin{aligned} \langle\cdot,\cdot\rangle_C : \ell_2 \times \ell_2 \to (\mathbb{R}^n, \mathcal{P}, \|\cdot\|) \\ \text{by } \langle x, y\rangle_C = \textstyle\sum_{k=1}^{n} e_k \langle x, y\rangle, \end{aligned} \tag{1}$$

then $\langle\cdot,\cdot\rangle_C$ is a S-cone inner product for ℓ_2-space.

Proof. We will show that $\langle\cdot,\cdot\rangle_C$ in (1) satisfies the following properties:

(IC1) Since $\langle x, x\rangle \geq 0$, then $\langle x, x\rangle_C = \sum_{k=1}^{n} e_k \langle x, x\rangle_C \succcurlyeq \theta$ for every $x \in \ell_2$; Furthermore, $\langle x, x\rangle_C = \sum_{k=1}^{n} e_k \langle x, x\rangle = \theta$ if and only if $\langle x, x\rangle =$ if and only if $x = 0$;
(IC2) $\langle x, y\rangle_C \underline{\quad = \quad} \sum_{k=1}^{n} e_k \langle x, y\rangle = \sum_{k=1}^{n} e_k \langle y, x\rangle = \underline{\langle y, x\rangle_C}\ \overline{\langle y, x\rangle}_C$ for every $x, y \in \ell_2$. Therefore $\langle x, y\rangle_C = \overline{\langle y, x\rangle}_C$;
(IC3) $\langle \alpha x, y\rangle_C = \sum_{k=1}^{n} e_k \langle \alpha x, y\rangle = \alpha \sum_{k=1}^{n} e_k \langle x, y\rangle = \alpha \langle x, y\rangle_C$ for every $x, y \in \ell_2$ and $\alpha \in \mathbb{R}$;
(IC4) For every $x, y, z \in \ell_2$ by triangle inequality of the inner product, we have

$$\begin{aligned} \langle x+y, z\rangle_C &= \sum_{k=1}^{n} e_k \langle x+y, z\rangle \\ &= \sum_{k=1}^{n} e_k (\langle x, z\rangle + \langle y, z\rangle) \\ &= \sum_{k=1}^{n} e_k \langle x, z\rangle + \sum_{k=1}^{n} e_k \langle y, z\rangle \\ &= \langle x, z\rangle_c + \langle y, z\rangle_C. \end{aligned}$$

Therefore, we can conclude that the function $\langle\cdot,\cdot\rangle_C$ in (1) is a S-cone inner product for ℓ_2-space. ☐

In an inner product space, vectors x and x are orthogonal if and only if $\langle x, y\rangle = 0$. We define orthogonality in S-cone inner product spaces analogously to those in the inner product space, which is given by the following theorems.

Theorem 2. *Let $(\ell_2, \langle\cdot,\cdot\rangle)$ be an inner product space with $\mathcal{P}$ is a S-cone, and if we define a S-cone inner product $\langle\cdot,\cdot\rangle_C : \ell_2 \times \ell_2 \to (\mathbb{R}, \mathcal{P}, \|\cdot\|)$ by $\langle x, y\rangle_C = \sum_{k=1}^{n} e_k \langle x, y\rangle$, then two vectors x and y are orthogonal if and only if $\langle x, y\rangle_C = \theta$.*

Proof. Since vectors x and y are orthogonal in an inner product space, i.e. $\langle x, y\rangle = 0$, we have

$$\langle x, y\rangle_C = \sum_{k=1}^{n} e_k \langle x, y\rangle = \sum_{k=1}^{n} e_k \cdot 0 = \theta.$$

On the other hand, if $\langle x, y\rangle_C = \theta$, then we get $\langle x, y\rangle_C = \sum_{k=1}^{n} e_k \langle x, y\rangle = \theta$. This result implies that $\langle x, y\rangle = 0$. ☐

Theorem 3. *Let $(\ell_2, \langle\cdot,\cdot\rangle)$ be an inner product space and the S-cone inner product on ℓ_2-space is defined by $\langle x, y\rangle_C = \sum_{k=1}^{n} e_k \langle x, y\rangle$, then*

(i) $\langle x, x\rangle_C = \|x\|_C \|x\|_C$*;*
(ii) $\langle x, y\rangle_C \preccurlyeq \|x\|_C \|\|_C$.

Proof.

(i) Since $\langle x, y\rangle_C = \sum_{k=1}^{n} e_k \langle x, y\rangle$, we have that $\langle x, x\rangle_C = \sum_{k=1}^{n} e_k \langle x, x\rangle = \sum_{k=1}^{n} e_k \|x\|^2 = \sum_{k=1}^{n} e_k \|x\| \|x\|$. Therefore, the multiplication $\|x\|_C \|x\|_C = \sum_{k=1}^{n} e_k \|x\| \|x\|$, and it means that $\langle x, x\rangle_C = \sum_{k=1}^{n} e_k \|x\| \|x\| = \|x\|_C \|x\|_C$.
(ii) By triangle inequality of the inner product, we have

$$\begin{aligned} \langle x, y\rangle_C^2 &= \sum_{k=1}^{n} e_k \langle x, y\rangle^2 \preccurlyeq \sum_{k=1}^{n} \langle x, x\rangle \langle y, y\rangle \\ &= \sum_{k=1}^{n} e_k (\|x\|^2 \|y\|^2) = \|x\|_C^2 \|y\|_C^2 \\ &= (\|x\|_C \|y\|_C)^2. \end{aligned}$$

Thus, we have that $\langle x, y\rangle_C \preccurlyeq \|x\|_c \|y\|_C$. ☐

Theorem 4. *Let $(\ell_2, \langle\cdot,\cdot\rangle)$ be an inner product space and the S-cone inner product on ℓ_2-space is defined by $\langle x,y\rangle_C = \sum_{k=1}^n e_k \langle x,y\rangle$, then*

(i) $\langle x,y\rangle_C + \langle w,z\rangle_C = \sum_{k=1}^n e_k(\langle x,y\rangle + \langle w,z\rangle)$.

(ii) $\langle \alpha x,y\rangle_C = \alpha \langle x,y\rangle_C$.

(iii) *If φ is an angle between vectors x and y in ℓ_2-space, then $\langle x,y\rangle_C = \|x\|_C \|y\|_C \cos\varphi$.*

Proof.

(i) $\langle x,y\rangle_C + \langle w,z\rangle_C = \sum_{k=1}^n e_k \langle x,y\rangle + \sum_{k=1}^n e_k \langle w,z\rangle = \sum_{k=1}^n e_k(\langle x,y\rangle + \langle w,z\rangle)$.

(ii) $\langle \alpha x,y\rangle_C = \sum_{k=1}^n e_k \langle \alpha x,y\rangle = \sum_{k=1}^n e_k\alpha \langle x,y\rangle = \alpha \sum_{k=1}^n e_k \langle x,y\rangle = \alpha \langle x,y\rangle_C$.

(iii) Since $\langle x,y\rangle = \|x\|\,\|y\|\cos\varphi$, then

$$\langle x,y\rangle_C = \sum_{k=1}^n e_k \langle x,y\rangle = \sum_{k=1}^n e_k \|x\|\,\|y\|\cos\varphi$$
$$= \cos\varphi \sum_{k=1}^n e_k \|x\|\,\|y\| = \|x\|_C \|y\|_C \cos\varphi.$$

Therefore, we get $\langle x,y\rangle_C = \|x\|_C \|y\|_C \cos\varphi$. □

From the discussion of the cone norm and S-cone inner product, we obtain its properties, among others: additive, multiplication with a scalar and multiplication between two cones. Furthermore, we construct and define a cone 2-norm associated with the S-cone inner product.

Let ℓ_2-space be a 2-normed space. A function $\|\cdot,\cdot\|_C : \ell^2 \times \ell_2 \to (\mathbb{R}^n, P, \|\cdot\|)$, be defined by $\|x,y\|_C = \sum_{k=1}^n e_k \|x,y\|_{\ell_2}$, is a cone 2-normed space. In this case, we call ℓ_2-space as cone 2-normed spaces. The reason for the name can be explained as follows.

For every $x,y,z \in \ell_2$ and $\alpha \in \mathbb{R}$, the following statements hold:

(CN1) $\|x,y\|_C = \sum_{k=1}^n e_k \|x,y\|_{\ell_2} \succcurlyeq \theta$ for all $x,y \in X$, because $\|x,y\|_{\ell_2} \geq 0$.

(CN2) $\|x,y\|_C = \sum_{k=1}^n e_k \|x,y\|_{\ell_2} = \theta$ if and only if $\|x,y\|_{\ell_2} = 0$ as 2-normed space, then $\|x,y\|_{\ell_2} = 0$ if and only if x and y are linearly dependent.

(CN3) $\|x,y\|_C = \sum_{k=1}^n e_k \|x,y\|_{\ell_2} = \sum_{k=1}^n e_k \|y,x\|_{\ell_2} = \|y,x\|_C$.

(CN4) Since $\|x,y+z\| \leq \|x,y\| + \|y,z\|$, then

$$\|x,y+z\|_C = \sum_{k=1}^n e_k \|x,y+z\|_{\ell_2}$$
$$\preccurlyeq \sum_{k=1}^n \left(\|x,y\|_{\ell_2} + \|x,z\|_{\ell_2}\right)$$
$$= \sum_{k=1}^n e_k \|x,y\|_{\ell_2} + \sum_{k=1}^n \|x,z\|_{\ell_2}$$
$$= \|x,y\|_C + \|x,z\|_C .$$

Which means that $\langle x+y,z\rangle_C \preccurlyeq \langle z,z\rangle_C + \langle y,z\rangle_C$. Therefore, an ℓ_2-space is a cone 2-normed space.

Theorem 5. *Let $(\ell_2, \langle\cdot,\cdot\rangle)$ be a S-cone inner product on ℓ_2-space, then*

$$\|x,y\|_C^2 = \sum_{k=1}^n e_k \|x,y\|_{\ell_2} = \|x\|_C^2 \|y\|_C^2 - (\langle x,y\rangle_C)^2.$$

And also

$$\|x,y\|_C^2 = \begin{vmatrix} \langle x,x\rangle_C & \langle x,y\rangle_C \\ \langle y,x\rangle_C & \langle y,y\rangle_C \end{vmatrix}$$

Proof. From the definition of the $\|\cdot,\cdot\|_C$, we have

$$(\|x,y\|_C)^2 = \sum_{k=1}^n e_k(\|x,y\|_{\ell_2})^2 = \sum_{k=1}^n e_k \begin{vmatrix} \langle x,x\rangle & \langle x,y\rangle \\ \langle y,x\rangle & \langle y,y\rangle \end{vmatrix}$$
$$= \sum_{k=1}^n e_k[\langle x,x\rangle \langle y,y\rangle - (\langle x,y\rangle)^2]$$
$$= \sum_{k=1}^n e_k \langle x,x\rangle \langle y,y\rangle - \sum_{k=1}^n e_k(\langle x,y\rangle)^2$$
$$= \langle x,x\rangle_C \langle y,y\rangle_C - (\langle x,y\rangle_C)^2$$
$$= \|x\|_C^2 \|y\|_C^2 - (\langle x,y\rangle_C)^2.$$

Therefore, we have

$$\|x,y\|_C^2 = \sum_{k=1}^n e_k \|x,y\|_{\ell_2} = \|x\|_C^2 \|y\|_C^2 - (\langle x,y\rangle_C)^2.$$

In addition,

$$(\langle x,y\rangle_C)^2 = \sum_{k=1}^n e_k \langle x,x\rangle \langle y,y\rangle - \sum_{k=1}^n e_k \langle x,y\rangle^2$$
$$= \sum_{k=1}^n e_k \langle x,x\rangle \langle y,y\rangle - \sum_{k=1}^n e_k \langle x,y\rangle \langle y,x\rangle$$
$$= \|x\|_C^2 \|y\|_C^2 - \langle x,y\rangle_C \langle y,x\rangle_C .$$

Since $\|x\|_C^2 = \|x,x\|_C$ and $\|y\|_C^2 = \langle y,y\rangle_C$ then

$$\|x,y\|_C^2 = \|x\|_C^2 \|y\|_C^2 - \langle x,y\rangle_C \langle y,x\rangle_C$$
$$= \langle x,x\rangle_C \langle y,y\rangle_C - \langle x,y\rangle_C \langle y,x\rangle_C .$$

Here we have $\|x,y\|_C^2 = \begin{vmatrix} \langle x,x\rangle_C & \langle x,y\rangle_C \\ \langle y,x\rangle_C & \langle y,y\rangle_C \end{vmatrix}$.

Example 1. Let $(\ell_2, \langle\cdot,\cdot\rangle)$ be an inner product space with $\mathcal{P}$ a S-cone in $\mathbb{R}^2$. If we define a function

$$\langle\cdot,\cdot\rangle_C : \ell_2 \times \ell_2 \to (\mathbb{R}^2, \mathcal{P}, \|\cdot\|)$$
$$\text{by } \langle x,y\rangle_C = (\langle x,y\rangle, \langle x,y\rangle), \quad (2)$$

then $\langle\cdot,\cdot\rangle_C$ is a S-cone inner product in ℓ_2-space.

We show that $\langle\cdot,\cdot\rangle_C$ satisfies the following properties:

(IC1) Since $\langle x,x\rangle \geq 0$, then $\langle x,x\rangle_C = (\langle x,x\rangle, \langle x,x\rangle) \succcurlyeq \theta$ for every $x \in \ell_2$. Furthermore, $\langle x,x\rangle_C = (\langle x,x\rangle, \langle x,x\rangle) = \theta$ if and ony if $\langle x,x\rangle = 0$ if and only if $x = 0$.

(IC2) $\langle x,y\rangle_C = (\langle x,y\rangle, \langle y,x\rangle) = \left(\overline{\langle y,x\rangle}, \overline{\langle y,x\rangle}\right) = \overline{\langle y,x\rangle}_C$ for every $x,y \in \ell_2$.
Thus, we have $\langle x,y\rangle_C = \overline{\langle y,x\rangle}_C$.

(IC3) From the property of multiplication by scalar, we have

$$\langle \alpha x,y\rangle_C = (\langle \alpha x,y\rangle, \langle \alpha x,y\rangle) = \alpha(\langle x,y\rangle, \langle x,y\rangle)$$
$$= \alpha \langle x,y\rangle_C$$
$$\text{for every } x,y \in \ell_2 \quad \text{and } \alpha \in \mathbb{R}.$$

(IC4) Using triangle inequality of innter product, for every $x,y,z \in \ell_2$, we have

$$\langle x+y,z\rangle_C = (\langle x+y,z\rangle, \langle x+y,z\rangle)$$

$$\begin{aligned}
&\preccurlyeq (\langle x,z\rangle + \langle y,z\rangle, \langle x,z\rangle + \langle y,z\rangle)\\
&= (\langle x,z\rangle, \langle x,z\rangle) + (\langle y,z\rangle, \langle y,z\rangle)\\
&= \langle x,z\rangle_C + \langle y,z\rangle_C
\end{aligned}$$

Then, we conclude that $\langle\cdot,\cdot\rangle$ in (2) is a S-cone inner product in ℓ_2.

Example 2. Let $(\ell_2, \|\cdot,\cdot\|)$ be a standard 2-norm space and $(\ell_2, \langle\cdot,\cdot\rangle)$is a S-cone inner product space as in Example 1. A function $\|\cdot,\cdot\|_C : \ell_2 \times \ell_2 \to (\mathbb{R}^2, P, \|\cdot\|)$ defined by

$$\|x,y\|_C = \big(\, \|x,y\|_{\ell_2}, \|x,y\|_{\ell_2}\big) \tag{3}$$

is a cone 2-norm in ℓ_2-space.

We see that it is defined as the standard norms:

$$\begin{aligned}
\|x,y\|^2 &= \begin{vmatrix}\langle x,x\rangle & \langle x,y\rangle\\ \langle y,x\rangle & \langle y,y\rangle\end{vmatrix} = \langle x,x\rangle\langle y,y\rangle - \langle x,y\rangle^2\\
&= \|x\|^2\|y\|^2 - \langle x,y\rangle^2,
\end{aligned}$$

and implies that

$$\begin{aligned}
&(\|x,y\|_C)^2\, \|x,y\|_C\, \|x,y\|_C\\
&= (\|x,y\|_{\ell_2}, \|x,y\|_{\ell_2}) \cdot (\|x,y\|_{\ell_2}, \|x,y\|_{\ell_2})\\
&= \big(\, \|x,y\|_{\ell_2}^2, \|x,y\|_{\ell_2}^2\big)\\
&= (\langle x,x\rangle\langle y,y\rangle - \langle x,y\rangle^2, \langle x,x\rangle\langle y,y\rangle - \langle x,y\rangle^2)\\
&= (\langle x,x\rangle\langle y,y\rangle - \langle x,y\rangle\langle y,x\rangle, \langle x,x\rangle\langle y,y\rangle - \langle x,y\rangle\langle y,x\rangle)\\
&= (\langle x,x\rangle\langle y,y\rangle, \langle x,x\rangle\langle y,y\rangle) - (\langle x,y\rangle\langle y,x\rangle, \langle x,y\rangle\langle y,x\rangle)\\
&= \langle x,x\rangle_C\langle y,y\rangle_C - \langle x,y\rangle_C\langle y,x\rangle_C.
\end{aligned}$$

Therefore $\|x,y\|_C^2 = \langle x,x\rangle_C\langle y,y\rangle_C - \langle x,y\rangle_C\langle y,x\rangle_C$. In other word, it means that $\|x,y\|_C^2 = \begin{vmatrix}\langle x,x\rangle_C & \langle x,y\rangle_C\\ \langle y,x\rangle_C & \langle y,y\rangle_C\end{vmatrix}$.

Now, we arrive at the main result of this paper, formulated in the following theorem.

Theorem 6. *Let $\|x,y\|_C = \sum_{k=1}^n e_k \|x,y\|_{\ell_2}$ be a cone 2-norm and φ an angle between vectors x and y in ℓ_2-space, then*

$$\|x,y\|_C^2 = (1-\cos^2\varphi)\, \|x\|_C^2\, \|y\|_C^2.$$

Proof. Since $\|x,y\|_C = \sum_{k=1}^n e_k \|x,y\|_{\ell_2}$ is a cone 2-norm, then we have $\|x,y\|_C^2 = \begin{vmatrix}\langle x,x\rangle_C & \langle x,y\rangle_C\\ \langle y,x\rangle_C & \langle y,y\rangle_C\end{vmatrix}$. It means that

$$\begin{aligned}
\|x,y\|_C^2 &= \|x\|_C^2\|y\|_C^2 = \langle x,y\rangle_C\langle y,x\rangle_C\\
&= \sum_{k=1}^n e_k\|x\|^2\|y\|^2 - \sum_{k=1}^n e_k\langle x,y\rangle\langle y,x\rangle\\
&= \sum_{k=1}^n e_k\big(\, \|x\|^2\, \|\|^2 - \langle x,y\rangle^2\big)\\
&= (1-\cos^2\varphi)\langle x,x\rangle_C\langle y,y\rangle_C\\
&= (1-\cos^2\varphi)\, \|x\|_C^2\, \|y\|_C^2.
\end{aligned}$$

Corollary 1. *et $\|x,y\|_C = \sum_{k=1}^n e_k \|x,y\|_{\ell_2}$ be a cone 2-norm and φ an angle between vectors x and y in ℓ_2-space. Then $\|x,y\|_C = \|x\|_C\|y\|_C \sin\varphi$.*

In a S-cone inner product space, two vectors x and y are orthogonal if and only if $\langle x,y\rangle_C = \theta$. It implies that $\langle x,y\rangle_C = \|x\|_C\|y\|_C\cos\varphi = \theta$, and also $\|x,y\|_C^2 = \|x\|_C^2\|y\|_C^2 - \langle x,y\rangle_C^2$, and we have $\|x,y\|_C^2 = \|x\|_C^2\|y\|_C^2 - \theta = \|x\|_C^2\|y\|_C^2$. As a conclusion, we have $\|x,y\|_C = \|x\|_C\|y\|_C$.

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