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Bankruptcy Probability Modeling Using Integro-Differential Equations With Gamma Distributed Claims

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Abstract

This study aims to model and analyze the ruin probability of an insurance company using an integro-differential equation under the assumption that claim sizes follow a Gamma distribution. The research method employed is a literature study, conducted by reviewing relevant books and scientific articles. The modeling process is carried out using an integro-differential equation and simplifying it into the form of a homogeneous linear differential equation. Numerical computations are performed using Python programming, and the results are presented in tables and graphs to facilitate analysis. The model is developed in four cases based on variations in the parameters of the Gamma distribution. The results show that the ruin probability decreases as the initial capital and premium loading increase, and increases as the expected value of claims rises. Therefore, an increase in the insurance company's surplus will reduce the risk of ruin.

Keywords: Insurance, Ruin Probability, Integro-Differential Equation, Gamma Distribution

1. Introduction

Insurance means coverage or protection of an object from danger that can cause loss (Fauzi, 2021: 13). Legally, insurance in Indonesia has been outlined in the Commercial Law Code (KUHD) Chapter 9 Article 246, which states that "insurance or coverage is an agreement, in which the insurer binds himself to the insured by receiving a premium, to provide him with compensation for a loss, damage, or not getting the expected profit, which may be suffered due to an uncertain event." The agreement that has been made between the policyholder and the insurance company is known as an insurance policy, allowing someone to protect themselves from the risks they experience.

Policyholders are required to pay premiums periodically, and the insurance company will compensate for the risks incurred through payment of a sum of money as stated in the policy, known as an insurance claim. This premium is the insured's responsibility in exchange for the protection the company provides them (Kismawadi et al., 2021). If the total claims submitted exceed the company's reserves, the company could potentially experience losses and even bankruptcy. Bankruptcy in insurance companies can be assessed using the concept of ruin probability, which is the chance that a company's financial reserves will fall below zero, making it unable to cover claim payment obligations (Dwiyawara & Widodo, 2022).

The probability of bankruptcy can be modeled using an integro-differential equation that provides an explicit relationship between the probability of bankruptcy, the rate of claim arrival, and the nature of the distribution of claim amounts, so that the calculation depends not only on the expected value of claims but also on the full structure of the distribution (Rolski et al., 1999: 118). In this study, the distribution of claim amounts used is the Gamma distribution. This distribution was chosen because it has two flexible parameters, namely the shape and scale parameters. These parameters can describe claim data that tend to be right-skewed, so that it can be seen that small claims occur more frequently, but there is still the possibility of large claims appearing.

Previous research by Vremiro et al., (2016) discussed the probability of bankruptcy with a Gamma distribution of claim amounts, which concluded that the greater the initial capital value of an insurance company, the smaller the probability of bankruptcy. Another study conducted by Maulida et al., (2025) who modeled and analyzed the probability of bankruptcy of insurance companies using an integral differential equation with the assumption that the claim amount is distributed as a combination of two exponential distributions. The results showed that the probability of bankruptcy is inversely proportional to the initial capital and premium loading, while the probability of bankruptcy is proportional to the expected value of claims.

Based on this study, the integro-differential equation approach has proven relevant in modeling the probability of insurance company bankruptcy. Therefore, the author aims to model and analyze the probability of insurance company bankruptcy using integro-differential equations, assuming that the claim size follows a Gamma distribution.

The results of this study are expected to explain the influence of claim distribution parameters on bankruptcy probability and contribute to insurance risk management studies.

2. Research Method

2.1 Compound Poisson Surplus Process

The surplus process is mathematically formulated as the difference between total premium income and total claims paid up to time t. The surplus model is written as

$$U(t) = u + ct - S(t); \qquad u \geq 0, c > 0, t \geq 0,$$

with

$U(t)$: surplus at time t;
u : initial surplus;
c : rate of premium receipts per unit of time (*premium rate*).

If the surplus is negative for the first time, then a bankruptcy claim can be said to have occurred (Bowers et al., 1997: 400). Premiums are assumed to be paid continuously at a constant rate c per unit time. The cost burden on the total premium in the time interval $[0, t]$ is stated as ct and interest is ignored to simplify the calculation. Furthermore, in this case it is assumed that the premium has a positive loading, namely $ct > E[S(t)]$, which results in $c > \lambda\mu$. If an additional premium of $\theta\lambda\mu$ is taken, then we obtain

$$c = (1 + \theta)\lambda\mu$$

or

$$\frac{\lambda}{c} = \frac{1}{(1+\theta)\mu} = \frac{1}{(1+\theta)E[X]}; \qquad \theta > 0, \tag{1}$$

with θ is an additional percentage of the premium (*premium loading*).

2.2 Probability of Bankruptcy

The probability of bankruptcy is the probability of an insurance company's surplus becoming negative for the first time. According to Bowers et al. (1997: 400), in the classical risk model, bankruptcy occurs when the total cumulative claims paid up to a certain point exceed the total funds available from initial capital and premiums received. Mathematically, the time to first bankruptcy can be expressed as

$$T = min\, t : t \geq 0, U(t) < 0.$$

If $T = \infty$, then the company will not experience bankruptcy, whereas if $T < \infty$, bankruptcy occurs at time $T = t$. The probability of bankruptcy for an infinite time with an initial surplus of u is expressed as

$$\begin{aligned}\psi(u) &= P(T < \infty)\\ &= P(\, U(t) < 0,\ t \geq 0 \mid U(0) = u\,)\end{aligned}$$

which describes the probability that a company with initial capital u will go bankrupt at some time $t \geq 0$. Furthermore, for a finite time horizon, the probability of bankruptcy is defined as

$$\psi(u, t) = P(T < t); \qquad 0 \leq t \leq T$$

which states that the probability of bankruptcy occurring before time t.

The probability of bankruptcy has a complementary relationship with the probability of survival. The probability of survival is the probability that an insurance company with initial capital u will never go bankrupt during its operating period. This probability is denoted by $\phi(u)$ and is mathematically defined as

$$\phi(u) = P(\, U(t) \geq 0,\ t \geq 0 \mid U(0) = u\,).$$

The relationship between these two opportunities is expressed by

$$\phi(u) = 1 - \psi(u),$$

so that the probability of bankruptcy in an unlimited time is expressed by

$$\psi(u) = 1 - \phi(u). \tag{2}$$

2.3 Gamma Distribution

According to Hogg et al., (1997: 173), the Gamma distribution is a continuous distribution that is often used to model positive random variables, such as the size of insurance claims. This distribution has two parameters, namely the shape parameter $\alpha > 0$ and the scale parameter $\beta > 0$, which determine the shape and spread of the distribution. A random variable X is said to follow the Gamma distribution with parameters α and β if it has a probability density function

$$f(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)\,\beta^{\alpha}}\, x^{\alpha-1}\, e^{-\frac{x}{\beta}}; & x > 0 \\ 0; & x \leq 0 \end{cases} \tag{3}$$

The distribution in Equation (3) is denoted by $X \sim$ Gamma (α, β), where $\Gamma(\alpha)$ denotes the Gamma function, which is defined for all $\alpha > 0$ as

$$\Gamma(\alpha) = \int x^{\alpha-1} e^{-x}\, dx.$$

The special case when α is a positive integer greater than 1, then

$$\Gamma(\alpha) = (\alpha - 1)!.$$

This shows that the Gamma function is a generalization of the factorial function on positive real numbers. Then, the expected value of the Gamma distribution is given by

$$E[X] = \alpha\beta. \tag{4}$$

2.4 Integral-Differential Equations

An integro-differential equation is a form of equation that combines the derivative and integral of a function (Dwiyawara & Widodo, 2022). In this study, an integro-differential equation is used to model the probability of bankruptcy in the face of unexpected claim risks, considering the rate of change of initial capital and the premium rate. This equation is chosen because it allows for analytical solutions that facilitate bankruptcy probability analysis. Two theorems related to the probability of survival from bankruptcy are given, which form the basis for deriving the bankruptcy probability model.

Theorem 2.4 (Klugman *et al.*, 2008: 288)

The probability of surviving bankruptcy (survival probability) when the company has no initial capital u is stated as

$$\phi(0) = \frac{\theta}{1+\theta}.$$

Theorem 2.5 (Klugman *et al.*, 2008: 288)

The chance of surviving bankruptcy (survival) with initial capital u meets

$$\begin{aligned}\phi'(u) &= \frac{\lambda}{c}\phi(u) - \frac{\lambda}{c}\int_0^u \phi(u-x)\, dF(x)\\ &= \frac{\lambda}{c}\phi(u) - \frac{\lambda}{c}\int_0^u \phi(u-x)\, f(x)\, dx\end{aligned}$$

This study uses Theorem 2.5, an integro-differential equation, as the basis for developing a bankruptcy probability model in Chapter 4. The steps for using an integro-differential equation to determine the probability of bankruptcy are as follows:

1. Determine the probability density function of the distribution of the number of claims used and substitute it into the integro-differential equation.
2. Simplify the obtained equation by performing differentiation so that it can be transformed into a homogeneous linear differential equation and its characteristic equation can be determined.
3. Find the roots of the characteristic equation to obtain a general solution and use this to determine the probability of survival from bankruptcy, which is then formulated into a bankruptcy probability model.

2.5 Linear Differential Equations

According to Zill (2018: 4), an ordinary differential equation (PDB) is defined as an equation that only contains ordinary derivatives of one or more dependent variables with respect to one independent variable. An n-order GDP is called linear if the equation is linear regarding the dependent variable z and all its derivatives, namely $z', z'', \dots, z^{(n)}$. The linearity of a GDP can be seen from its nature, namely the dependent variable and all its derivatives have the highest power of one, and the coefficients only depend on the independent variables. The general form of an n-order linear GDP is stated as (Zill, 2018: 5)

$$a_n(u)z^{(n)} + a_{n-1}(u)z^{(n-1)} + \cdots + a_1(u)z' + a_0(u)z = g(u) \tag{5}$$

with $a_0(u), a_1(u), \dots, a_n(u)$ are the coefficients that depend most on the independent variable u and $a_0(u) \neq 0$. Equation (2.17) is said to be homogeneous if $g(u) = 0$, whereas if $g(u) \neq 0$ then equation (2.17) is said to be non-homogeneous.

According to Sihwaningrum & Dasril (2021: 30), the general form of homogeneous linear GDP of order-n with constant coefficients is

$$a_n(u)z^{(n)} + a_{n-1}(u)z^{(n-1)} + \cdots + a_1(u)z' + a_0(u)z = 0. \tag{6}$$

Suppose the solution to Equation (6) is $z = e^{pu}$, so that

$$\left.\begin{aligned} z' &= pe^{pu} \\ z'' &= p^2 e^{pu} \\ &\vdots \\ z^{(n)} &= p^n e^{pu} \end{aligned}\right\} \qquad \cdots \tag{7}$$

Next, if Equation (7) is substituted into Equation (6), then

$$[a_n(u)p^n + a_{n-1}(u)p^{n-1} + \cdots + a_1(u)p + a_0(u)]e^{pu} = 0. \tag{8}$$

Since $e^{pu} \neq 0$, the characteristic equation is obtained

$$a_n(u)p^n + a_{n-1}(u)p^{n-1} + \cdots + a_1(u)p + a_0(u) = 0. \tag{9}$$

If the roots $p_1, p_2, \dots, p_n$ of the characteristic equation are real and distinct, then the general solution of the homogeneous linear differential equation is

$$z(u) = A_1 e^{p_1 u} + A_2 e^{p_2 u} + \cdots + A_n e^{p_n u} \tag{10}$$

with $A_1, A_2, \dots, A_n$ is an arbitrary constant

3. Results and Discussion

3.1 Derivation of Gamma-Distributed Integro-Differential Equations$(\boldsymbol{\alpha}, \boldsymbol{\beta})$

Based on (2), the probability of surviving bankruptcy needs to be constructed first by substituting the probability density function of the Gamma distribution in Equation (3) into the integro-differential equation in Theorem 2.5, which yields

$$\phi'1(u) = \frac{\lambda}{c}\phi(u) - \frac{\lambda}{c}\int_0^u \phi(u-x)\frac{1}{\Gamma(\alpha)\,\beta^{\alpha}}\, x^{\alpha-1}\, e^{-\frac{x}{\beta}}\, dx. \tag{11}$$

Furthermore, Equation (11) is called an integro-differential equation with a claim size distributed Gamma (α, β). When $\alpha = 1$, then $\Gamma(1) = (1-1)! = 0! = 1$ and $x^{1-1} = x^0 = 1$, so based on equation (11) we obtain

$$\phi'(u) = \frac{\lambda}{c}\phi(u) - \frac{\lambda}{c}\int_0^u \phi(u-x)\frac{1}{\beta}e^{-\frac{x}{\beta}}\, dx. \tag{12}$$

Suppose $y = u - x$, then $dy = -dx$ or $dx = -dy$. As a result, the limits of the integral change, when $x = 0$, then $y = u$ and $x = u$ when $x = u$, then $y = 0$. Thus, Equation (12) can be written as

$$\phi'(u) = \frac{\lambda}{c}\phi(u) - \frac{\lambda}{c}\int_u^0 \phi(y)\frac{1}{\beta}e^{-\frac{(u-y)}{\beta}}(-dy). \tag{13}$$

Then, by using the integral property that $\int_u^0(-dy) = \int_0^u dy$, we obtain an integro-differential equation with a large claim distribution Gamma $(1, \beta)$, namely

$$\phi'(u) = \frac{\lambda}{c}\phi(u) - \frac{\lambda}{c\beta}e^{-\frac{u}{\beta}}\int_0^u \phi(y)e^{\frac{y}{\beta}}\, dy. \tag{14}$$

Next, to determine the integro-differential equation with the claim size distributed Gamma when $\alpha = 2$ and $\alpha = 3$ the same steps are carried out as when $\alpha = 1$. The integro-differential equation with the claim size distributed Gamma $(1, \beta)$ and Gamma $(2, \beta)$ respectively is

$$\phi'(u) = \frac{\lambda}{c}\phi(u) - \frac{\lambda u}{c\beta^2}e^{-\frac{u}{\beta}}\int_0^u \phi(y)e^{\frac{y}{\beta}}\, dy + \frac{\lambda}{c\beta^2}e^{-\frac{u}{\beta}}\int_0^u \phi(y)ye^{\frac{y}{\beta}}\, dy \tag{15}$$

And

$$\phi'(u) = \frac{\lambda}{c}\phi(u) - \frac{\lambda u^2}{2c\beta^3}e^{-\frac{u}{\beta}}\int_0^u \phi(y)e^{\frac{y}{\beta}}\, dy - \frac{\lambda u}{2c\beta^3}e^{-\frac{u}{\beta}}\int_0^u \phi(y)y^2e^{\frac{y}{\beta}}\, dy + \frac{\lambda u}{c\beta^3}e^{-\frac{u}{\beta}}\int_0^u \phi(y)ye^{\frac{y}{\beta}}\, dy. \tag{16}$$

3.2 Solving Gamma-Distributed Integro-Differential Equations$(\boldsymbol{\alpha}, \boldsymbol{\beta})$

In Equation (14) there is still an integral form, so it will be differentiated with respect to u to eliminate the integral form to become

$$\phi''(u) = \frac{\lambda}{c}\phi'(u) + \frac{\lambda}{c\beta}e^{-\frac{u}{\beta}}\int_0^u \phi(y)e^{\frac{y}{\beta}}\, dy - \frac{\lambda}{c\beta}\phi(u). \tag{17}$$

Ifboth sides of Equation (14) are multiplied by $\frac{1}{\beta}$, then

$$\frac{\lambda}{c\beta^2}e^{-\frac{u}{\beta}}\int_0^u \phi(y)e^{\frac{y}{\beta}}\, dy = \frac{\lambda}{c\beta}\phi(u) - \frac{1}{\beta}\phi'(u). \tag{18}$$

Then, by substituting Equation (18) into Equation (17), we obtain

$$\phi''(u) + \frac{1}{\beta}\phi'(u) - \frac{\lambda}{c}\phi'(u) = 0. \tag{19}$$

Based on Equation (4), $E[X] = \alpha\beta = (1)\beta = \beta$, so Equation (1) can be written as,

$$\frac{\lambda}{c} = \frac{1}{(1+\theta)\beta}. \tag{20}$$

Therefore, by substituting Equation (20) into Equation (19) we obtain

$$\phi''(u) + \frac{\theta}{(1+\theta)\beta}\phi'(u) = 0. \tag{21}$$

Furthermore, by assuming $z = \phi'(u)$, Equation (21) can be expressed as a first-order homogeneous linear differential equation, namely

$$z' + \frac{\theta}{(1+\theta)\beta}z = 0. \tag{22}$$

The characteristic equation of Equation (22) is

$$p + \frac{\theta}{(1+\theta)\beta} = 0.$$
$$\Leftrightarrow p = -\frac{\theta}{(1+\theta)\beta}.$$

Based on Equation (10), the general solution to Equation (22) is expressed as

$$\phi'(u) = K_1 e^{pu}. \tag{23}$$

Suppose $u = 0$, then Equation (23) becomes $\phi'(0) = K_1$. Then, considering Theorem 2.4 and Theorem 2.5 when $u = 0$, we obtain

$$\phi'(0) = \frac{\lambda}{c}\phi(0)$$
$$\phi'(0) = \frac{1}{(1+\theta)\beta}\left(\frac{\theta}{1+\theta}\right)$$
$$\phi'(0) = \frac{\theta}{(1+\theta)^2\beta}$$

so that

$$K_1 = \frac{\theta}{(1+\theta)^2\beta}.$$

Next, by integrating Equation (23) with respect to$u$, obtained

$$\begin{aligned} \phi(u) &= \int K_1 e^{pu} du \\ \phi(0) &= \frac{K_1}{p}e^{pu} + K_2 \end{aligned} \tag{24}$$

Chances of surviving bankruptcy when$u$approaching infinity is 1, so we get $K_2 = 1$. As a result, Equation (24) can be written as

$$\phi(u) = \frac{K_1}{p}e^{pu} + 1. \tag{25}$$

The bankruptcy probability model can be formulated by substituting Equation (25) into Equation (3) to become

$$\psi(u) = 1 - \phi(u)$$
$$\phi(0) = 1 - \left(\frac{K_1}{p}e^{pu} + 1\right)$$
$$\phi(0) = -\frac{\left(\frac{\theta}{(1+\theta)^2\beta}\right)}{\left(-\frac{\theta}{(1+\theta)\beta}\right)}e^{-\frac{\theta}{(1+\theta)\beta}u}$$
$$\phi(0) = \frac{1}{1+\theta}e^{-\frac{\theta}{(1+\theta)\beta}u}.$$

Thus, the bankruptcy probability model using an integro-differential equation with a large claim distribution Gamma $(1, \beta)$ is

$$\psi(u) = \frac{1}{1+\theta}e^{-\frac{\theta}{(1+\theta)\beta}u}. \tag{26}$$

Next, to determine the bankruptcy probability model using the integro-differential equation with the claim size being distributed Gamma when $\alpha = 2$ and $\alpha = 2$ the same steps are carried out as when $\alpha = 1$. The bankruptcy probability model using the integro-differential equation with the claim size being distributed Gamma $(2, \beta)$ is

$$\psi(u) = -\frac{A_1}{p_1}e^{p_1 u} - \frac{A_2}{p_2}e^{p_2 u} \tag{27}$$

with

$$p_1 = \frac{-3 - 4\theta + \sqrt{9 + 8\theta}}{(1+\theta)4\beta};$$
$$p_2 = \frac{-3 - 4\theta - \sqrt{9 + 8\theta}}{(1+\theta)4\beta};$$
$$A_1 = \frac{1}{p_1 - p_2}\left(\frac{p_1 p_2}{1+\theta} + \frac{\theta p_1}{(1+\theta)^2 2\beta}\right);$$
$$A_2 = \frac{\theta}{(1+\theta)^2 2\beta} - \frac{1}{p_1 - p_2}\left(\frac{p_1 p_2}{1+\theta} + \frac{\theta p_1}{(1+\theta)^2 2\beta}\right).$$

The bankruptcy probability model uses an integro-differential equation with a large claim distribution Gamma $(3, \beta)$ is

$$\psi(u) = -\frac{L_1}{p_1}e^{p_1 u} - \frac{L_2}{p_2}e^{p_2 u} - \frac{L_3}{p_3}e^{p_3 u} \tag{28}$$

with

$$p_1 = \frac{1}{6}\sqrt[3]{j} + \frac{20 - 18\theta}{\sqrt[3]{j}(1+\theta)^2 27\beta^2} - \frac{8 + 9\theta}{(1+\theta)9\beta};$$
$$p_2 = \frac{-k + \sqrt{k^2 - 4s}}{2};$$
$$p_3 = \frac{-k - \sqrt{k^2 - 4s}}{2};$$
$$L_1 = \frac{\theta}{(1+\theta)^2 3\beta} - L_2 - L_3;$$
$$L_2 = \left(\frac{3\beta\theta(1+\theta)p_1 - \theta}{(1+\theta)^3 9\beta^2 p_1} - \left(\frac{p_1 - p_3}{p_1}\right)L_3\right)\left(\frac{p_1}{p_1 - p_2}\right);$$
$$L_3 = \left(\frac{3\beta\theta(1+\theta)p_1 - \theta}{(1+\theta)^3 9\beta^2 p_1} - \frac{3\beta\theta(1+\theta)p_1 p_2 - \theta p_2}{(1+\theta)^2 3\beta p_1} + p_2\right)\left(\frac{p_1 p_3}{(p_3 - p_1)(p_2 - p_3)}\right);$$
$$j = \frac{1088 + 2052\theta + 972\theta^2}{(1+\theta)^3 27\beta^3} + \frac{4\sqrt{96 + 176\theta + 81\theta^2}}{(1+\theta)^2\beta^3};$$
$$k = \frac{8 + 9\theta}{(1+\theta)9\beta} + p_1;$$
$$s = -\frac{\theta}{(1+\theta)\beta^3 p_1}.$$

3.3 Bankruptcy Probability Model Analysis

Based on the bankruptcy probability model using the integro-differential equation with the claim size obtained with Gamma distribution, namely in Equations (26), (27), and (28), it can be analyzed that the bankruptcy probability $(\psi(u))$ is influenced by several factors, namely the initial capital of the insurance company (u), the percentage of additional premium or premium loading (θ), and the expected value of claims. The analysis was carried out on three bankruptcy probability models with parameters of the form $\alpha = 1, 2, and$ 3.

Based on Equation (26), the bankruptcy probability model decreases exponentially with respect to initial capital. When u→∞, then the value of $e^{-\frac{\theta}{(1+\theta)\beta}u} \to 0$, so that $\psi(u) \to 0$. In other words, increasing initial capital causes the bankruptcy probability to approach zero. Increasing the value of θ causes the exponential coefficient to become smaller, so that the bankruptcy probability decreases. In addition, the parameter β is directly related to the expected claim value, that is, if the value of the parameter β is greater, the expected claim value will also be greater. When β→∞, then the value of $e^{-\frac{\theta}{(1+\theta)\beta}u} \to e^0$. This causes the rate of premium receipts to become slower, so that the bankruptcy probability tends to increase.

Based on Equation (27), it is known that p_1 and p_2 are roots that satisfy $p_2 < p_1 < 0$. If $u \to \infty$, then $e^{p_1 u} \to 0$ and $e^{p_2 u} \to 0$. This condition causes the value of $\psi(u) \to 0$, so that the probability of bankruptcy decreases. Furthermore, an increase in the value of θ causes the values of p_1 and p_2 to become smaller, so that the values of A_1 and A_2 become smaller and the probability of bankruptcy decreases. Conversely, if the expected value of claims is greater $(\beta \to \infty)$, then the values of p_1 and p_2 will become smaller. As a result, the rate of premium receipts becomes slower, so that the probability of bankruptcy increases.

Based on Equation (28), it is known that p_1, p_2, and p_3 are roots that satisfy $p_3 < p_2 < p_1 < 0$. If $u \to \infty$, then $e^{p_1 u} \to 0$, $\quad ^{p_2 u} \to 0$ and $e^{p_6 u} \to 0$. As a result, the value of $\psi(u) \to 0$, so the probability of bankruptcy is smaller. An increase in the value of θ will cause the values of p_1, p_2, and p_3 to become smaller and followed by a decrease in the values of L_1, L_2, and L_3. This condition results in a decreasing probability of bankruptcy. Conversely, if the expected value of claims is greater $(\beta \to \infty)$, then the values of p_1, p_2, and p_3 will be closer to zero. This causes the rate of premium receipt to be slower, so the probability of bankruptcy increases.

Based on this analysis, it can be concluded that:

if the initial capital is bigger, the probability of bankruptcy $(\psi(u))$ is smaller;

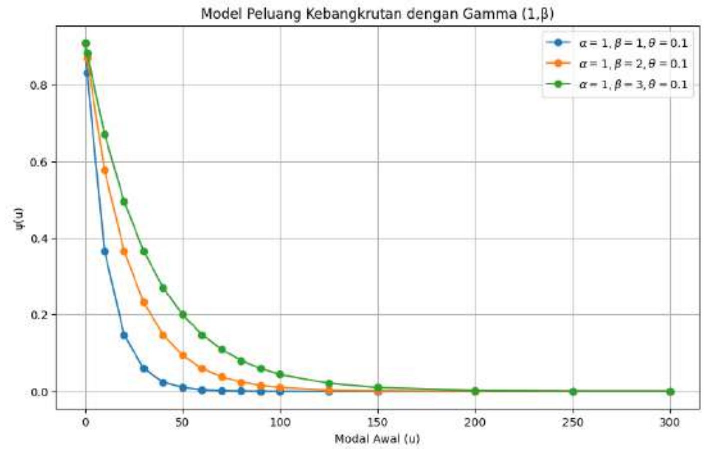
if the additional premium percentage or premium loading (θ) is greater, then the probability of bankruptcy $(\psi(u))$ is smaller;

If the expected claim value is greater, then the probability of bankruptcy $(\psi(u))$ is greater.

3.3 Application of the Bankruptcy Probability Model

The application of the bankruptcy probability model using integro-differential equations with Gamma distributed claim sizes for parameters $\alpha = 1$, 2, and 3 is carried out using the Python programming language to carry out numerical calculations and graph creation. The specified values are $\beta = 1$, 2, and 3 with $\theta = 0.1$, 0.2 and 0.3, and initial capital of $0, 1, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 125, 150, 200, 250, and\ 300$ units.

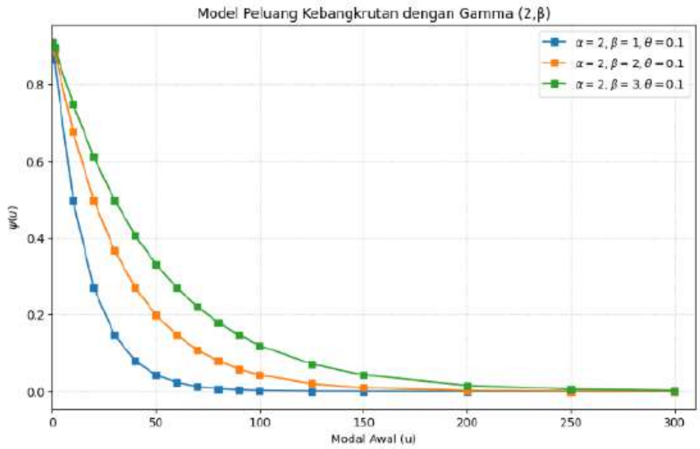
Case 1: Gamma-Distributed $(\mathbf{1}, \boldsymbol{\beta})$ Bankruptcy Probability Model



Picture 1 Bankruptcy Probability Model Graph with Gamma$(1, \beta)$

Figure 1 shows the relationship between the probability of bankruptcy and initial capital when the parameter $\alpha = 1$, while the parameter β changes. When the initial capital $u = 0$, it can be seen that $\psi(u)_{\beta=1} = \psi(u)_{\beta=2} = \psi(u)_{\beta=3}$. When $u \geq 1, \psi(u)_{\beta=1} < \psi(u)_{\beta=2} < \psi(u)_{\beta=3}$. This shows that when the initial capital is large enough, an increase in the value of β, which represents an increase in the expected value of claims, still increases the probability of bankruptcy, even if the value is small. In this study, the optimal initial capital for $\beta = 1{,}2$, and 3 with the criteria $\psi(u) < 0.05$, respectively, is 40, 70, and 100. Thus, the greater the initial capital of the insurance company, the weaker the influence of the β parameter on the probability of bankruptcy.

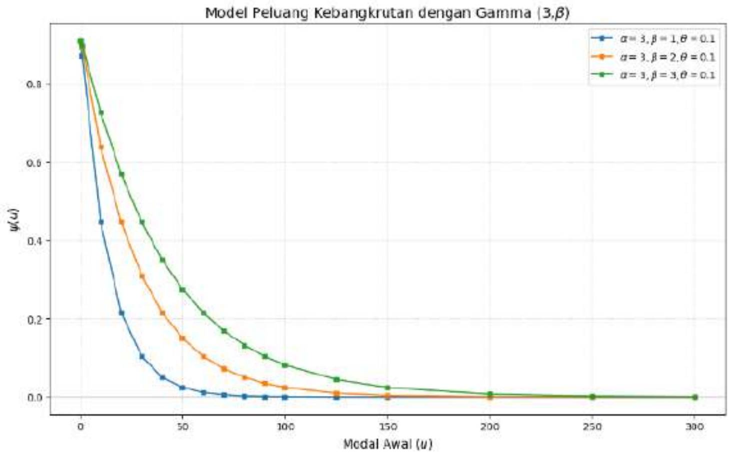
Case 2: Gamma-Distributed $(\mathbf{2}, \boldsymbol{\beta})$ Bankruptcy Probability Model



Picture 2 *:* Bankruptcy Probability Model Graph with Gamma$(2, \beta)$

Figure 2 shows the relationship between the probability of bankruptcy and initial capital when the parameter $\alpha = 2$, while the parameter β changes. When the initial capital $u = 0$, it can be seen that $\psi(u)_{\beta=1} = \psi(u)_{\beta=2} = \psi(u)_{\beta=3}$. In this study, with the criterion $\psi(u) < 0.05$, the optimal initial capital for $\beta = 1, 2, and\ 3\ are\ 50, 100, and\ 150$, respectively. When the initial capital is very large, the probability of bankruptcy for each value of β is very small. When the initial capital $u \geq 1$, $\psi(u)_{\beta=1} < \psi(u)_{\beta=2} < \psi(u)_{\beta=3}$, this is in line with the analysis of the probability density function of the Gamma distribution.

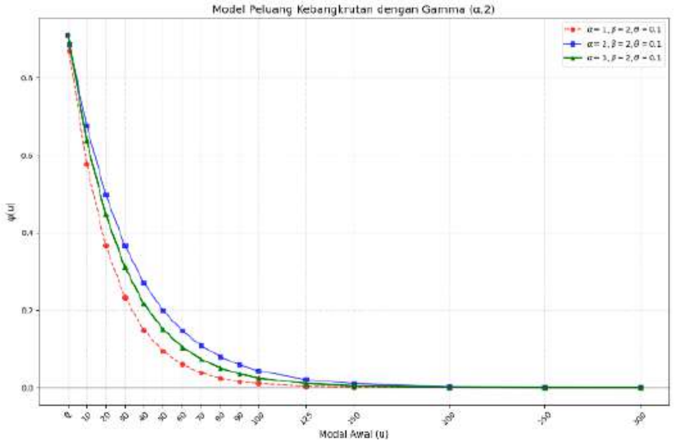
Case 3: Gamma-Distributed Bankruptcy Probability Model$(\mathbf{3}, \boldsymbol{\beta})$



Picture 3 : Bankruptcy Probability Model Graph with Gamma$(3, \beta)$

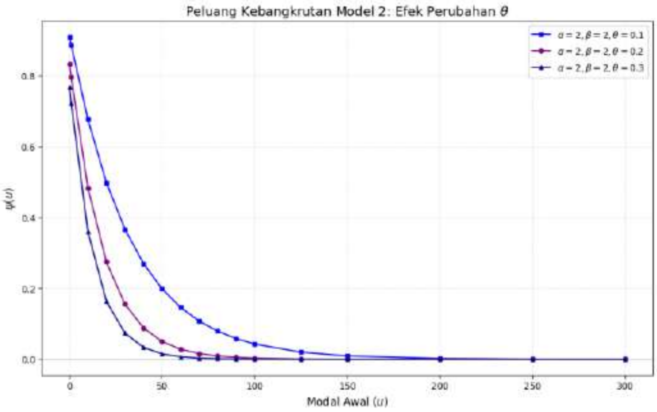
Figure 3 shows the relationship between the probability of bankruptcy and initial capital when the parameters and parameters change. When the initial capital$\alpha = 3\beta u$= 0 for every valueβ, the probability of bankruptcy is the same. In this study, the optimal initial capital for and with the criteria are respectively 50, 90, and 125. At very large initial capital, the probability of bankruptcy for each value$\beta = 1{,}2{,}3\psi(u) < 0{,}05\beta$is very small. When the initial capital is , , this is in line with the analysis of the probability density function of the Gamma distribution. Thus, the expected value of claims increases the risk of bankruptcy, but when the initial capital is large enough, the probability of bankruptcy decreases.$u \geq 1\psi(u)_{\beta=1} < \psi(u)_{\beta=2} < \psi(u)_{\beta=3}$

Case 4: Gamma-Distributed Bankruptcy Probability and Changing Value$(\boldsymbol{\alpha}, \mathbf{2})\boldsymbol{\theta}$



Picture 4 *:* Bankruptcy Probability Graph with Gamma$(\alpha, 2)$

Figure 4 shows the relationship between the probability of bankruptcy and initial capital when the parameter $\beta = 2$, while the parameter α changes. Based on Figure 4, the probability of bankruptcy decreases as the initial capital increases. When $u = 0$, then $\psi(u)_{\alpha=1} = \psi(u)_{\alpha=2} = \psi(u)_{\alpha=3}$. Then, when u>1, what happens is $\psi(u)_{\alpha=1} < \psi(u)_{\alpha=3} < \psi(u)_{\alpha=2}$, so it is not in line with the analysis of the probability density function of the Gamma distribution. This occurs because the probability of bankruptcy is not only influenced by the shape of the Gamma distribution, but also by the interaction of the claim parameter with the initial capital and premium in the model.



Picture 5 Bankruptcy Probability Graph with Changing Valuesθ

Figure 5 shows the relationship between the probability of bankruptcy and initial capital when the claim distribution parameters are fixed, namely $\alpha = 2$ and $\beta = 2$, while the premium loading (θ) changes, namely $\theta = 0.1,\ 0.2, and\ 0.3$. Based on Figure 5, it can be seen that the probability of bankruptcy decreases as the premium loading value increases. Therefore, $\psi(u)_{\theta=0.3} < \psi(u)_{\theta=0.2} < \psi(u)_{\theta=0.1}$ is obtained. This is because a larger premium loading value indicates that the premium received by the insurance company is higher than the expected claim value, so the company has a larger reserve fund to cover claims and bankruptcy risks.

Based on the four cases that have been analyzed, it can be concluded that the probability of bankruptcy decreases as the initial capital increases ($(\psi(u))u$), the greater the premium loading, the lower the probability of bankruptcy, and the greater the expected value of claims is directly proportional to the probability of bankruptcy ($(\theta)\psi(u)$), namely if the expected value of the claim increases (α And β The larger the value), the greater the probability of bankruptcy. This is in line with research by Vremiro et al. (2016) which concluded that the probability of bankruptcy decreases as initial capital increases. In addition, research by Maulida et al. (2025) also shows that the probability of bankruptcy is inversely proportional to initial capital and premium loading, and directly proportional to the expected value of claims. Thus, although this study uses variations in the parameters of the Gamma distribution form, the pattern of decreasing chances of bankruptcy with increasing initial capital remains consistent with previous research.

5. CONCLUSION

Based on the results and discussion in Chapter 4, the following conclusions were obtained.

1. The insurance company bankruptcy probability model uses an integro-differential equation with a Gamma distribution of claim amounts, namely
   - Gamma-Distributed $(1, \beta)$ Bankruptcy Probability Model
   $$\psi(u) = \frac{1}{1+\theta} e^{-\frac{\theta}{(1+\theta)\beta}u}$$
   - Gamma-Distributed $(2, \beta)$ Bankruptcy Probability Model
   $$\psi(u) = -\frac{A_1}{p_1} e^{p_1 u} - \frac{A_2}{p_2} e^{p_2 u}$$
   with
   $$p_1 = \frac{-3 - 4\theta + \sqrt{9 + 8\theta}}{(1+\theta)4\beta};$$
   $$p_2 = \frac{-3 - 4\theta - \sqrt{9 + 8\theta}}{(1+\theta)4\beta};$$
   $$A_1 = \frac{1}{p_1 - p_2}\left(\frac{p_1 p_2}{1+\theta} + \frac{\theta p_1}{(1+\theta)^2 2\beta}\right);$$
   $$A_2 = \frac{\theta}{(1+\theta)^2 2\beta} - \frac{1}{p_1 - p_2}\left(\frac{p_1 p_2}{1+\theta} + \frac{\theta p_1}{(1+\theta)^2 2\beta}\right).$$
   - Gamma-Distributed $(3, \beta)$ Bankruptcy Probability Model

$$\psi(u) = -\frac{L_1}{p_1}e^{p_1 u} - \frac{L_2}{p_2}e^{p_2 u} - \frac{L_3}{p_3}e^{p_3 u}$$

with

$$p_1 = \frac{1}{6}\sqrt[3]{j} + \frac{20-18\theta}{\sqrt[3]{j}(1+\theta)^2 27\beta^2} - \frac{8+9\theta}{(1+\theta)9\beta};$$

$$p_2 = \frac{-k+\sqrt{k^2-4s}}{2};$$

$$p_3 = \frac{-k-\sqrt{k^2-4s}}{2};$$

$$L_1 = \frac{\theta}{(1+\theta)^2 3\beta} - L_2 - L_3;$$

$$L_2 = \left(\frac{3\beta\theta(1+\theta)p_1-\theta}{(1+\theta)^3 9\beta^2 p_1} - \left(\frac{p_1-p_3}{p_1}\right)L_3\right)\left(\frac{p_1}{p_1-p_2}\right);$$

$$L_3 = \left(\frac{3\beta\theta(1+\theta)p_1-\theta}{(1+\theta)^3 9\beta^2 p_1} - \frac{3\beta\theta(1+\theta)p_1p_2-\theta p_2}{(1+\theta)^2 3\beta p_1} + p_2\right)\left(\frac{p_1p_3}{(p_3-p_1)(p_2-p_3)}\right);$$

$$j = \frac{1088+2052\theta+972\theta^2}{(1+\theta)^3 27\beta^3} + \frac{4\sqrt{96+176\theta+81\theta^2}}{(1+\theta)^2\beta^3};$$

$$k = \frac{8+9\theta}{(1+\theta)9\beta} + p_1;$$

$$s = -\frac{\theta}{(1+\theta)\beta^3 p_1}.$$

2. The research results show that the probability of bankruptcy decreases with increasing initial capital and premium loading, and increases with increasing expected claim value. In other words, the probability of bankruptcy is inversely proportional to initial capital and premium loading, and directly proportional to expected claim value. Therefore, maintaining a higher surplus can help insurance companies reduce their risk of bankruptcy

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