

# Connectivity of The Triple Idempotent Graph of Ring $\mathbb{Z}_n$

Vika Yugi Kurniawan, Bayu Purboutomo, Sutrima, Nughthoh Arfawi Kurdhi

**Abstract**—Let  $R$  be a commutative ring and  $I(R)$  denotes a set of all idempotent elements of  $R$ . The triple idempotent graph of ring  $R$ , denoted by  $TI(R)$ , is the undirected simple graph with vertex-set in  $R - \{0, 1\}$ . Two distinct vertices  $u$  and  $v$  in  $TI(\mathbb{Z}_n)$  are adjacent if and only if there exists  $w \in R - \{0, 1\}$  where  $w \neq u$  and  $w \neq v$  such as  $uw \notin I(R)$ ,  $vw \notin I(R)$ ,  $vw \notin I(R)$  and  $uvw \in I(R)$ . In this research, we study the connectivity of the triple idempotent graph of ring integer modulo  $n$ , denoted by  $TI(\mathbb{Z}_n)$ . The result is that the triple idempotent graph of ring  $\mathbb{Z}_n$  is a connected graph if  $n$  prime and  $n \geq 7$ .

**Index Terms**—the triple idempotent graph, ring of integers modulo  $n$ , connected graph

## I. INTRODUCTION

LET  $R$  be a commutative ring with unit element  $1 \neq 0$ . In 1988, Beck [4] introduced the concept of a zero-divisor graph that connect between ring theory and graph theory. In [2], Anderson and Livingston modified zero-divisor graph, denotes by  $\Gamma(R)$ , with vertices  $Z(R)^* = Z(R) - \{0\}$  and two distinct vertices  $x, y \in Z(R)^*$  adjacent if and only if  $xy = 0$ . There was shown that  $\Gamma(R)$  is a connected graph with  $\text{gr}(\Gamma(R)) \in \{3, 4, \infty\}$ . In other paper, Akhtar and Lee [3], studied the connectivity of the zero divisor graph  $\Gamma(R)$  associated to a finite commutative ring  $R$ . They investigated the conditions of ring  $R$  such that graph  $\Gamma(R)$  is a connected graph. Later, many papers that investigated various kind of graphs associated with the ring, see [1], [10], [8], and [11].

Recently in [9], Mohammad and Shuker introduced graph that is called idempotent divisor graph, denoted by  $JI(R)$ , with the set of vertices  $R^* = R - \{0\}$  and two distinct vertices  $v_1$  and  $v_2$  adjacent if and only if  $v_1 \cdot v_2 = e$ , for some non-unit idempotent element  $e \in R$  (i.e.  $e^2 = e \neq 1$ ). Let  $I(R)$  be a set of idempotent elements of ring  $R$ . In this paper, the definition of the triple idempotent graph of a commutative ring  $R$ , denoted by  $TI(R)$ , is the undirected simple graph with vertex-set  $R - \{0, 1\}$ . Two distinct vertices  $u$  and  $v$  are in  $TI(R)$  adjacent if and only if there exist  $w \in R - \{0, 1\}$  where  $w \neq u$  and  $w \neq v$  such as  $uw \notin I(R)$ ,  $vw \notin I(R)$ ,  $vw \notin I(R)$  and  $uvw \in I(R)$ . We will investigate the properties that related to connectivity of the triple idempotent graph of ring integer modulo  $n$ , denoted by  $TI(\mathbb{Z}_n)$ .

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## II. PRELIMINARIES

According to Chartrand and Zhang [7], a graph  $G$  is a finite nonempty set  $V$  of objects is called vertices together with a possibly empty set  $E$  of 2-element subsets of  $V$  is called edges. The number of vertices in a graph  $G$  is the order of  $G$  and the number of edges is the size of  $G$ . A graph of size 0 is called an empty graph. Two distinct vertices  $u$  and  $v$  said to be adjacent if there is an edge between  $u$  and  $v$ . The degree of a vertex  $u$  in a graph  $G$  is the number of vertices in  $G$  that are adjacent to  $u$ . If a path between two vertices of graph  $G$  can be found, then the graph  $G$  is connected.

If  $R$  is a ring,  $Z(R)$  denotes the set of zero-divisors of  $R$  and  $I(R)$  denotes the set of idempotent elements of  $R$ .

**Definition 1.** Graph triple idempotent of commutative ring  $R$ , denoted by  $TI(R)$ , is the undirected simple graph with vertex-set  $R - \{0, 1\}$ , and two different vertices  $u$  and  $v$  are in  $TI(R)$  adjacent if and only if there exists  $w \in R - \{0, 1\}$  where  $w \neq u$  and  $w \neq v$  such that  $u \cdot v \notin I(R)$ ,  $u \cdot w \notin I(R)$ ,  $v \cdot w \notin I(R)$  and  $u \cdot v \cdot w \in I(R)$ , where  $I(R)$  is a set of all idempotent elements of  $R$ .

In the following, given an example of  $TI(\mathbb{Z}_9)$ .

**Example 2.** Let  $\mathbb{Z}_9$ , with  $\mathbb{Z}_9 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}\}$  and  $I(\mathbb{Z}_9) = \{\bar{0}, \bar{1}\}$ . By the Definition 1, the set of vertex  $V(TI(\mathbb{Z}_9)) = \{\bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}\}$  and the set of edge  $E(TI(\mathbb{Z}_9)) = \{e_{\bar{2}, \bar{4}}, e_{\bar{2}, \bar{8}}, e_{\bar{4}, \bar{8}}, e_{\bar{3}, \bar{7}}, e_{\bar{5}, \bar{8}}, e_{\bar{7}, \bar{8}}\}$ . Graph  $TI(\mathbb{Z}_9)$  illustrated in the Figure 1.

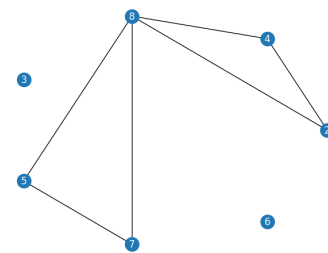


Fig. 1: Graph  $TI(\mathbb{Z}_9)$

## III. RESULT

In this section, the results of investigations regarding the conditions for connectivity of  $TI(\mathbb{Z}_n)$  are given. The following is a theorem regarding the condition for the  $TI(\mathbb{Z}_n)$  to be an empty graph.

**Theorem 3.** Let  $\mathbb{Z}_n$ . If  $n \leq 6$ , then  $TI(\mathbb{Z}_n)$  is an empty graph.

*Proof.* The proof given by 3 cases below.

1) For  $n = 3, 4$ .

Since  $|V(TI(\mathbb{Z}_n))| < 3$ , then there are not found any adjacency such that for  $TI(\mathbb{Z}_n)$  has no edge or  $TI(\mathbb{Z}_n)$  is an empty graph.

2) For  $n = 5$ .

There are found  $V(TI(\mathbb{Z}_5)) = \{\bar{2}, \bar{3}, \bar{4}\}$  and  $I(\mathbb{Z}_5) = \{\bar{0}, \bar{1}\}$ . Since  $|V(TI(\mathbb{Z}_5))| = 3$  and  $\mathbb{Z}_5$  is a commutative ring such that there is only one possible combination of vertices  $u, v, w$  i.e.  $u = \bar{2}, v = \bar{3}, w = \bar{4}$ . As a result of  $uv = \bar{1}, \bar{1} \in I(\mathbb{Z}_5)$ , then  $u, v, w$  are not adjacent. Therefore  $TI(\mathbb{Z}_5)$  has no edge or  $TI(\mathbb{Z}_5)$  is an empty graph.

3) For  $n = 6$ .

There are found  $V(TI(\mathbb{Z}_6)) = \{\bar{2}, \bar{3}, \bar{4}, \bar{5}\}$  and  $I(\mathbb{Z}_6) = \{\bar{0}, \bar{1}, \bar{3}, \bar{4}\}$ . Since  $|V(TI(\mathbb{Z}_6))| = 4$ , same as before, then there are four possible combinations of vertices  $u, v, w$ . First, for vertex  $u = \bar{2}, v = \bar{3}, w = \bar{4}$ . As result of  $uv = \bar{0} \in I(\mathbb{Z}_6)$ , then  $u, v, w$  are not adjacent. In the same way, for the others combination as well. Therefore,  $TI(\mathbb{Z}_6)$  has no edge or  $TI(\mathbb{Z}_6)$  is an empty graph.

So, it is proven that for  $\mathbb{Z}_n$  where  $n \leq 6$ , the  $TI(\mathbb{Z}_n)$  is an empty graph. ■

The illustrations of graph  $TI(\mathbb{Z}_n)$  where  $n \leq 6$  are shown in Figure 2.

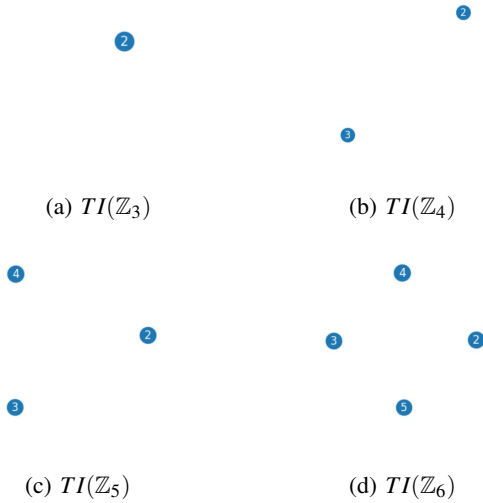


Fig. 2: Graph  $TI(\mathbb{Z}_n)$  where  $n \leq 6$

There is a lemma that related to divisibility properties for any non-zero element in a field  $F$ .

**Lemma 4.** Let  $F$  be a field. For every  $a, b \in F$  where  $a \neq 0$  and  $b \neq 0$  then  $a|b$  and  $b|a$ .

*Proof.* Let non zero element  $a, b \in F$ . We will show that  $a|b$  and  $b|a$ . Because of  $F$  is a field, clearly there exist  $b^{-1}$  such that  $a = a.e = a.b.b^{-1}$ . Using commutative and closed properties, then  $a.b.b^{-1} = a.b^{-1}.b = c.b$  with  $c = a.b^{-1}, c \in F$ . Therefore, surely  $b|a$ . As the same way, for  $a|b$ . So, for every non zero element  $a, b \in F$  then  $a|b$  and  $b|a$ . ■

In the following, a lemma is given regarding cases of vertices that are not adjacent to each other in  $TI(\mathbb{Z}_n)$ .

**Lemma 5.** Let  $\mathbb{Z}_n$  where  $n$  is prime and  $n \geq 7$ . For every  $u, v \in V(TI(\mathbb{Z}_n))$ ,  $u$  not adjacent to  $v$  if  $uv = 1$  or  $uv = x \neq 1$  where  $x = u^{-1}$  or  $x = v^{-1}$ .

*Proof.* Let  $u, v \in V(TI(\mathbb{Z}_n))$  be distinct and arbitrary vertices. Then, clearly that  $\mathbb{Z}_n$  where  $n$  prime are field, such that  $I(\mathbb{Z}_n) = \{0, 1\}$  where element 0 and 1 in  $\mathbb{Z}_n$  are related to  $\bar{0}$  and  $\bar{1}$ . Since, the field has no zero divisor element, then adjacency condition can be reduced to  $uv \neq 1, vw \neq 1, uw \neq 1$  and  $uvw = 1$ . There will be showed 2 cases where  $u$  is not adjacent to  $v$ .

1) Case 1  $uv = 1$ .

By Definition 1, if  $uv = 1, 1 \in I(\mathbb{Z}_n)$ , then it will be contradiction with one of adjacency conditions of  $TI(\mathbb{Z}_n)$ . Therefore,  $u$  is not adjacent to  $v$ .

2) Case 2  $uv = x \neq 1$ , where  $x = u^{-1}$  or  $x = v^{-1}$ .

For  $uv = u^{-1}$ , if both side multiply with  $u$ , then  $u.v.u = 1$ . Seen that needed two elements of  $u$  in the left side so that triple vertices multiplication that involved  $u$  and  $v$  is equal to 1. By Definition 1, it will be contradiction with one of adjacency conditions where  $u \neq v \neq w$  respectively. Therefore,  $u$  is not adjacent to  $v$ . In the same ways for  $uv = v^{-1}$ . ■

As an illustration of the lemma 5, the following is an example of an explanation of the cases that two vertices is not adjacent to each other in  $\mathbb{Z}_{11}$ .

**Example 6.** Let  $\mathbb{Z}_{11}$ . The set of vertex element and the set of idempotent element,  $V(TI(\mathbb{Z}_{11})) = \{\bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}, \bar{10}\}$  and  $I(TI(\mathbb{Z}_{11})) = \{\bar{0}, \bar{1}\}$ . We provide an example explanation by showing that any two vertices that are not adjacent in  $\mathbb{Z}_{11}$ , can be included in one of the two cases in the Lemma 5 above. Given below adjacency matrix of  $\mathbb{Z}_{11}$  in the Table I.

TABLE I: Adjacency Matrix of  $TI(\mathbb{Z}_{11})$

	2	3	4	5	6	7	8	9	10
2	0	0	1	1	0	1	1	1	1
3	0	0	0	0	1	1	1	0	1
4	1	0	0	0	0	1	1	0	1
5	1	0	0	0	1	1	0	0	1
6	0	1	0	1	0	1	1	1	1
7	1	1	1	1	1	0	0	0	1
8	1	1	1	0	1	0	0	1	1
9	1	0	0	0	1	0	1	0	1
10	1	1	1	1	1	1	1	1	0

Seen that  $\bar{2}$  is not adjacent to  $\bar{3}$ . This is because  $\bar{2}.\bar{3} = \bar{6} = \bar{2}^{-1}$  such that include in case 2. Then, vertex  $\bar{2}$  is also not adjacent to  $\bar{6}$  because  $\bar{2}.\bar{6} = \bar{1}$  such that include in case 1. Now, vertex  $\bar{3}$  is not adjacent to  $\bar{4}$  because  $\bar{3}.\bar{4} = \bar{1}$  such that include in case 1. Also, vertex  $\bar{3}$  is not adjacent to  $\bar{5}$  because  $\bar{3}.\bar{5} = \bar{4} = \bar{3}^{-1}$  such that include in case 2. In the same ways, for the others vertices that not adjacent each other in  $TI(\mathbb{Z}_{11})$  and always can be included to one of the two cases in Lemma 5. The  $TI(\mathbb{Z}_{11})$  is showed in Figure 3.

The following result show that for  $\mathbb{Z}_n$  where  $n$  prime and  $n \geq 7$ ,  $TI(\mathbb{Z}_n)$  is a connected graph.

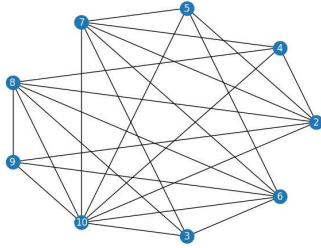


Fig. 3: Graph  $TI(\mathbb{Z}_{11})$

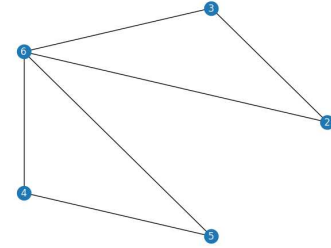


Fig. 4:  $TI(\mathbb{Z}_7)$

**Theorem 7.** Let  $\mathbb{Z}_n$ . If  $n$  prime and  $n \geq 7$ , then  $TI(\mathbb{Z}_n)$  is a connected graph.

*Proof.* Let  $u, v \in V(TI(\mathbb{Z}_n))$  be arbitrary. Then, by Lemma 5 adjacency condition can be reduced to  $uv \neq 1, vw \neq 1, uw \neq 1$  and  $uvw = 1$ . If  $u$  and  $v$  are adjacent, then there is a direct path  $u-v$ . If  $u$  and  $v$  are not adjacent, then by Lemma 5 there are divided to 2 cases.

- 1) Case 1. If  $uv = 1$ . Let  $w \in V(TI(\mathbb{Z}_n))$ . There exist  $w^{-1} \in V(TI(\mathbb{Z}_n))$  where  $w.w^{-1} = 1$ . Since  $u \in V(TI(\mathbb{Z}_n))$ , then by Lemma 4,  $w^{-1} = u.x$  such that  $w.u.x = w.w^{-1} = 1$ . Therefore,  $u$  adjacent to  $w$ . In the same way for  $v$ , such that  $v$  adjacent to  $w$ . Since  $u$  and  $v$  are adjacent to  $w$ , then there is a path  $u-w-v$ .
- 2) Case 2. If  $uv = x \neq 1$ , where  $x = u^{-1}$  or  $x = v^{-1}$ . Let  $uv = u^{-1}$ . Since  $u^{-1} \in V(TI(\mathbb{Z}_n))$ , then by Lemma 4 there exist  $w \in V(TI(\mathbb{Z}_n))$  such that  $u^{-1} = w.y$ . So,  $u.w.y = u.u^{-1} = 1$  such that  $u$  adjacent to  $w$ . Then, also for  $v^{-1} \in V(TI(\mathbb{Z}_n))$  by Lemma 4,  $v^{-1}$  can be divided by  $w$  such that  $v^{-1} = w.z$ . Therefore,  $v.w.z = v.v^{-1} = 1$ , then  $v$  adjacent to  $w$ . Since  $u$  adjacent to  $w$  and  $v$  adjacent to  $w$  then there is a path  $u-w-v$ . With same ways, can be proved for  $uv = v^{-1}$ .

Since there is always can be found a path between  $u$  and  $v$ , then  $TI(\mathbb{Z}_n)$  where  $n$  prime and  $n \geq 7$  is a connected graph. ■

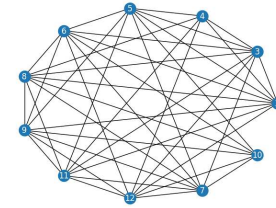
Given example that related to a connected graph of  $TI(\mathbb{Z}_7)$  as an illustration of Theorem 7.

**Example 8.** Let  $\mathbb{Z}_7$ . There is obtained vertex set  $V(TI(\mathbb{Z}_7)) = \{2, 3, 4, 5, 6\}$  and idempotent set  $I(\mathbb{Z}_7) = \{0, 1\}$ . Now, let  $u, v \in V(TI(\mathbb{Z}_7))$  be distinct vertices, will be shown there always exist path between  $u$  and  $v$ . For  $u = 2$  and  $v = 3$ , there exist  $w = 6$  such that  $u.v.w = \bar{1} \in I(\mathbb{Z}_7)$ . Therefore, there is a direct path between  $2-3, 2-6$  and  $3-6$ .

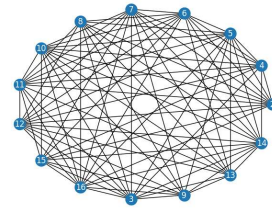
For  $u = \bar{2}$  and  $v = \bar{4}$ , as result of  $\bar{2}.\bar{4} = \bar{1}, \bar{1} \in I(\mathbb{Z}_7)$  then vertex  $\bar{2}$  and  $\bar{4}$  are not adjacent. So, there exist  $w = \bar{6}$  such that  $w^{-1} = \bar{6} = \bar{2}.\bar{3} = \bar{4}.\bar{5}$ . Therefore,  $\bar{2}$  and  $\bar{4}$  adjacent to  $\bar{6}$ . Can be found a path  $\bar{2}-\bar{6}-\bar{4}$ , as the same ways for  $u = \bar{3}$  and  $v = \bar{5}$ . For vertex  $u = \bar{2}$  and  $v = \bar{5}$ ,  $u.v = \bar{3}$  where  $\bar{3}$  is the inverse of vertex  $\bar{5}$ . There exist  $w = \bar{6}$  such that  $v^{-1} = \bar{3} = \bar{6}.\bar{4}$  and  $u^{-1} = \bar{4} = \bar{3}.\bar{6}$ . Then, clearly  $\bar{2}$  and  $\bar{5}$  adjacent to  $\bar{6}$  such that there is a path  $\bar{2}-\bar{6}-\bar{5}$  as the same ways for  $u = \bar{3}$  and  $v = \bar{4}$ .

Since there is always found a path between every two distinct vertices, then  $TI(\mathbb{Z}_7)$  is a connected graph. Figure 4 shows graph  $TI(\mathbb{Z}_7)$ .

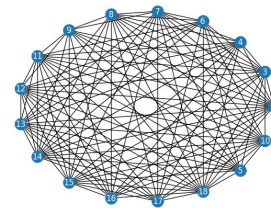
As shown in Figure 3,  $TI(\mathbb{Z}_{11})$  is a connected graph because it also satisfies Theorem 1. In addition, we also give several other graphs  $TI(\mathbb{Z}_n)$  where  $n$  is prime and  $n \geq 7$  in Figure 5.



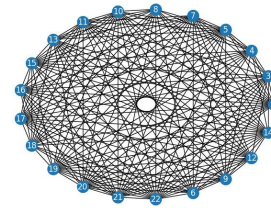
(a)  $TI(\mathbb{Z}_{13})$



(b)  $TI(\mathbb{Z}_{17})$



(c)  $TI(\mathbb{Z}_{19})$



(d)  $TI(\mathbb{Z}_{23})$

Fig. 5: Graph  $TI(\mathbb{Z}_n)$  where  $n = \{13, 17, 19, 23\}$ .

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## REFERENCES

- [1] Akbari, S., Kiani, D., Mohammadi, F. and Moradi, S.: The total graph and regular graph of a commutative ring. *Journal of pure and applied algebra*, 213(12), pp.2224-2228 (2019).
- [2] Anderson, D. F., Livingston, P.S.: The Zero-Divisor Graph of a Commutative Ring. *J. Algebra*, 217, pp: 434-447 (1999).
- [3] Akhtar, R. and Lee, L.: Connectivity of the zero-divisor graph for finite rings. *Involve, a Journal of Mathematics*, 9(3), pp.415-422 (2016).
- [4] Beck, I.: Coloring of Commutative Ring. *J. Algebra*, Vol. 116, No. 1, pp: 208-226 (1988).
- [5] Çelikel, E.C.: Triple Zero Graph of Commutative Ring. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, Vol. 70 (2), pp: 653-663 (2021).
- [6] Chartrand, G.: *Introductory Graphs Theory*. Dover Publication. Inc, New York (1997).
- [7] Chartrand, G., Lesniak L. and Zhang, P.: *Graphs & Digraphs*. Discrete Applied Mathematics. Dover Publication. Inc, New York (2012).
- [8] Mariiaa, H., Kurniawan, V.Y. and Sutrima. : Zero annihilator graph of semiring of matrices over Boolean semiring. *AIP Conference Proceedings*. Vol. 2326. No. 1 (2021).
- [9] Mohammad, H.Q. dan Shuker, N.H. : The Idempotent Divisor Graph of Commutative Ring. *Iraqi Journal*, Vol. 63, No. 2, pp: 645-651 (2022).
- [10] Osba, E.A., Al-Addasi, S. and Jaradeh, N.A.: Zero divisor graph for the ring of Gaussian integers modulo  $n$ . *Communications in Algebra*, 36(10), pp.3865-3877 (2008).
- [11] Rahmawati, A., Kurniawan, V.Y. and Wibowo, S.: The diameter of annihilator ideal graph of  $\mathbb{Z}_n$ . *AIP Conference Proceedings*. Vol. 2326. No. 1, pp.020020 (2021).