Transformation of the generalized chaotic system into canonical form

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ARTICLE INFO

Article history:
Received September 30, 2017
Revised October 11, 2017
Accepted October 11, 2017

Keywords:
Chaos
Dynamical system
Differential equations
Generalized chaotic system
Nonlinear coordinate transformation

A B S T R A C T

The paper deals with the development of a numerical algorithm for transforming a generalized chaotic system into its canonical form. Such transformation allows us to simplify control algorithm for chaotic system. This algorithm is defined by using Lie derivatives for output variable and a solution of nonlinear equations. The use of the proposed algorithm is one of the ways for discovering new chaotic attractors. These attractors can be obtained by transformation of known chaotic systems into various state spaces. Transformed attractors depend on both parameters of chaotic system and sample time of its discrete model.

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I. Introduction

The theory of chaotic systems is a fast-growing branch of the dynamic system theory. This branch has a wide application in various spheres of human activities, such as robotic [1], communication [2], cryptography [3], meteorology [4], economy or business application [5], and so on. Great interest to the chaotic systems was caused by their unique properties. Microcontroller one can use these sequences in various ways. For example, they can be used for setting up secure data transmission, planning path of mobile robot, investigating exchange rate fluctuations. This list can be continued for pages.

Wide ranges of applications of chaotic systems have caused a great number of its researches. One can find a lot of papers on researches on dynamics and implementations of integer-order [1]–[3] and fractional-order [6] chaotic systems in continuous-time and discrete-time domains. These researches proposed the novel chaotic systems [7] and investigated existing ones [1]–[3] [6].

One of the directions of the chaotic systems theory is control of chaotic systems. So many publications on chaos control [7][8] and chaos systems synchronization [2][3][9] can be found in scientific press today. The great interest to chaos control is caused by the possibility to test novel control algorithms for nonlinear unstable dynamical objects. If these algorithms work correctly for chaotic systems, they will work for various industrial objects with stable dynamics likewise.

The feedback linearization [10] is one of the effective control technique for nonlinear controller construction, but the main drawback of this linearization is the use of the object's complete state vector. This fact makes the researcher to set up and to use tons of different sensors. It is obvious that the control system becomes more complex and difficult to configure.

To avoid this drawback, we propose to transform a chaotic system's dynamic into a canonical form. It allows us to use only one sensor in the control system feedback. The transformation of the chaotic system into the canonical form is known only for one class of chaotic systems [11][12] and it is hard to use it for another one.

In this paper, we propose to perform transformation of an arbitrary chaotic system into a canonical form by using generalized approach based on differential geometry methods and nonlinear algebraic equations' solution. We suggest using numerical methods while the mentioned transformation is being performed. It avoids us to use complex mathematical apparatus and gives numerical algorithms, which can be used as numerical routines while control system is being programmed on microcontroller.

Our paper is organized as follows: firstly, we get a transformation procedure for a general dynamical object given in the continuous-time domain. We then adapt the mentioned procedure for discrete-time domain. Finally, we show usage of proposed approach for transformation continuous-time and discrete-time dynamics of Lorenz system into canonical form.

II. Method

A. Continuous-time Transformation Algorithm for A Generalized Dynamical Object

Let us consider a generalized n-th order continuous-time dynamical object given in the following way

$$\dot{x}_j = f_j(x_i), \quad i, j = 1,\ldots,n, \tag{1}$$

where x_i, x_j are state variables of dynamical object, $f_j(x_i)$ are some nonlinear functions.

We assume that these functions are differentiable in all state variables x_i for n times. This assumption allows us to transform (1) into canonical form

$$\begin{aligned} \dot{y}_j &= y_{j+1}; \quad j = 1,\ldots,n-1 \\ \dot{y}_n &= g_n(y_i), \end{aligned} \tag{2}$$

where $\dot{y}_i$ are new state variables, $g_n(y_i)$ are nonlinear functions.

One can perform the above mentioned nonlinear coordinate transformation by using the following algorithm:

1. One state variable x_k is selected as output variable

   $$y_1 = x_k; \quad k = 1,\ldots,n, \tag{3}$$

   where k is the number of output variable.
2. This variable is differentiated for n times and Lie derivatives are defined [10]:

   $$y_{i+1} = L_{\mathbf{f}}^{i} x_k; \quad i = 1,\ldots,n, \tag{4}$$

   where $\mathbf{f}$ is an (n x 1)-size matrix of functions $f_j(x_i)$

   $$\mathbf{f} = (f_1(x_i) \quad f_2(x_i) \quad \cdots \quad f_n(x_i))^T. \tag{5}$$
3. The interrelations between new y_i and old x_i state variables are defined as solution the first n-1 equations of (4) for x_i thus

   $$x_i = A(y_i), \quad i = 1,\ldots,n-1, \tag{6}$$

   where $A(y_i)$ is some nonlinear operator.
4. The unknown function $g_n(y_i)$ is defined from n-th equation of (4) by substituting into the Lie derivative $L_{\mathbf{f}}^{n} x_k$ (6).

The given algorithm allows us to get transformed equations of a nonlinear object given by (1) into a canonical form. The main drawback of the proposed method is the difficulty in analytically determining the $A(y_i)$-operator. This operator in the elementary functions can be defined only for the short range right-hand expressions in (1). The determination of the $A(y_i)$-operator is associated with the usage of non-elementary functions in general case. The definition of these functions is a separate nontrivial scientific problem with a weak practical usage due to the usage of complex mathematical apparatus.

We propose to simplify the determination of the $A(y_i)$-operator by transition into discrete-time domain and using numerical methods.

B. Discrete-time Transformation Algorithm for A Generalized Dynamical Object

The known numerical methods are based on various approximations of the differentiation operator. These approximations are built on the basis of future, current, and past values of state variables.

We use a following general approximation of differentiation operator [13]:

$$\dot{x} = dx/dt \approx d\left(x[i+q], x[i+q-1], \cdots, x[i], x[i-1], \cdots, x[i-w]\right), \quad (7)$$

where $x[i]$ is the value of state space variable x in i-th time interval, $x[i+q]$ is the value of state variable x on q-th time interval in the future, and $x[i-w]$ is the value of state x on w-th interval in the past; or in z-form:

$$\dot{x} \approx d\left(z^q x, z^{q-1} x, \cdots, x, z^{-1} x, \cdots, z^{-w} x\right), \quad (8)$$

where z^{-1} is the one step backward shift operator and z^1 is the one step forward shift operator.

An approximation for j-th order differential operator can be written down by using (8) in the following way:

$$\dot{x}^{(j)} \approx d^j\left(z^q x, z^{q-1} x, \cdots, x, z^{-1} x, \cdots, z^{-w} x\right), \quad 2q \geq j, 2w \geq j. \quad (9)$$

One can rewrite (4) by using (9) thus

$$d^{i+1}\left(z^q y_1, z^{q-1} y_1, \cdots, y_1, z^{-1} y_1, \cdots, z^{-w} y_1\right) = L_{\mathbf{f}}^i x_k; \quad i \in [1, n]. \quad (10)$$

Solution of (10) allows us to determine interrelations between the new coordinate y_1 and old one x_i. We propose to use for solution of these equations iterative numerical methods like Newton-Raphson method [14]. This method allows us to write down the following iterative expression for state variables:

$$x_i = z^{-1} x_i - \frac{\partial F_i}{\partial \dot{F}_i}, \quad i = 1, \ldots, n, \quad (11)$$

where

$$F_i\left(z^{-1} x_i, y_1\right) = d^{i+1}\left(z^q y_1, z^{q-1} y_1, \cdots, y_1, z^{-1} y_1, \cdots, z^{-w} y_1\right) - L_{\mathbf{f}}^i x_k; \quad i = 1, \ldots, n. \quad (10)$$

Function $g_n(y_i)$ can be defined by substituting (11) into Lie derivative $L_{\mathbf{f}}^n x_k$. This function is used while we are making the transformation of the differential equations (1) into algebraic ones:

$$\begin{aligned} &d^j\left(z^q y_1, z^{q-1} y_1, \cdots, y_1, z^{-1} y_1, \cdots, z^{-w} y_1\right) = y_j; \; j = 1, \ldots, n-1, \\ &d^n\left(z^q y_1, z^{q-1} y_1, \cdots, y_1, z^{-1} y_1, \cdots, z^{-w} y_1\right) = g_n(y_i). \end{aligned} \quad (12)$$

Numerical solution of (12) allows us to define canonical state variables y_j in general case.

III. Results and Discussion

Now we show two examples of using a proposed approach to transform the differential equations in normal form into canonical one.

We consider a well-known Lorenz system, which is given by the following equations [15]:

$$\begin{aligned} \dot{x}_1 &= -\sigma x_1 + \sigma x_2; \\ \dot{x}_2 &= \rho x_1 - x_2 - x_1 x_3; \\ \dot{x}_3 &= x_1 x_2 - \beta x_3, \end{aligned} \quad (12)$$

where σ, ρ, β are some coefficients and x_i are state variables.

Equations (12) describe nonlinear objects with chaotic dynamic. Let us transform (12) into the classical matrix form;

$$\dot{\mathbf{X}} = \mathbf{f}(\mathbf{X}), \tag{13}$$

where

$$\begin{aligned} &\mathbf{X} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^T; \\ &\mathbf{f}(\mathbf{X}) = \begin{pmatrix} -\sigma x_1 + \sigma x_2 & \rho x_1 - x_2 - x_1 x_3 & x_1 x_2 - \beta x_3 \end{pmatrix}. \end{aligned} \tag{14}$$

We consider transformations of (12) into the canonical form for x_1 state variables.

A. Analytical Transformation of The Lorenz Equations for x_1 Variable

After selecting x_1 variable as output, we use y_i as new state variables. The new state variables y_i are defined as Lie derivatives of the output variable x_1

$$\begin{aligned} y_1 &= x_1; \\ y_2 &= L_{\mathbf{f}} x_1; \\ y_3 &= L_{\mathbf{f}}^2 x_1, \end{aligned} \tag{15}$$

where

$$\begin{aligned} L_{\mathbf{f}} x_1 &= -\sigma x_1 + \sigma x_2; \\ L_{\mathbf{f}}^2 x_1 &= -\sigma(-\sigma x_1 + \sigma x_2) + \rho\sigma x_1 - \sigma x_2 - \sigma x_1 x_3. \end{aligned} \tag{16}$$

Let us substitute (16) into (15)

$$\begin{aligned} y_1 &= x_1; \\ y_2 &= -\sigma x_1 + \sigma x_2; \\ y_3 &= -\sigma(-\sigma x_1 + \sigma x_2) + \rho\sigma x_1 - \sigma x_2 - \sigma x_1 x_3 \end{aligned} \tag{17}$$

or

$$\begin{aligned} y_2 &= -\sigma y_1 + \sigma x_2; \\ y_3 &= \sigma(\rho + \sigma) y_1 - \sigma(\sigma + 1) x_2 - \sigma y_1 x_3 \end{aligned}. \tag{18}$$

We solve (18) for the variables x_2 and x_3

$$\begin{aligned} x_2 &= \frac{y_2}{\sigma} + y_1; \\ x_3 &= \rho - 1 - \left(1 + \frac{1}{\sigma}\right)\frac{y_2}{y_1} - \frac{y3}{\sigma y_1} \end{aligned}. \tag{19}$$

Now let us find the 3-rd Lie derivative for variables x_1

$$L_{\mathbf{f}}^3 x_1 = -\sigma x_1^2 x_2 + \sigma\big((\beta + 2\sigma + 1)x_3 - 2\rho\sigma - \sigma^2 - \rho\big)x_1 + \sigma\big(-x_3\sigma + \rho\sigma + \sigma^2 + \sigma + 1\big)x_2. \tag{20}$$

We define an unknown function $g_3(y_1, y_2, y_3)$ by substituting (19) into (20):

$$g_3(y_1, y_2, y_3) = -\sigma y_1^3 - y_1^2 y_2 + \beta\sigma(\rho - 1)y_1 - \beta y_2(\sigma + 1) - y_3(\beta + \sigma + 1) + \frac{y_2^2(\sigma + 1) + y_2 y_3}{y_1}. \tag{21}$$

Finally, we can write down the Lorenz system's dynamic in canonical form:

$$\begin{aligned} \dot{y}_1 &= y_2; \\ \dot{y}_2 &= y_3; \\ \dot{y}_3 &= -\sigma y_1^3 - y_1^2 y_2 + \beta\sigma(\rho - 1)y_1 - \beta y_2(\sigma + 1) - y_3(\beta + \sigma + 1) + \frac{y_2^2(\sigma + 1) + y_2 y_3}{y_1}. \end{aligned} \tag{22}$$

It is simple to transform 3-rd order system of differential equations (22) into one 3-rd order equation

$$\dddot{y}_1 = -\sigma y_1^3 - y_1^2\dot{y}_1 + \beta\sigma(\rho - 1)y_1 - \beta\dot{y}_1(\sigma + 1) - \ddot{y}_1(\beta + \sigma + 1) + \frac{\dot{y}_1^2(\sigma + 1) + \dot{y}_1\ddot{y}_1}{y_1}. \quad (23)$$

We call the equation as Lorenz equation in the canonical form and the corresponding dynamical system as a continuous-time canonical Lorenz system.

Analyzing (22)-(23) allows us to formulate the following statement:

Statement 1: Equations of nonlinear system's dynamic in canonical form are more complex than in normal one. Thus, contrary to linear systems, whose mathematical model is simpler in canonical state space, the transformation of a nonlinear system into another state space does not allow us to simplify it.

Numerical solutions of (12) (curve 1) and (22) (curve 2) are shown on Fig. 1.

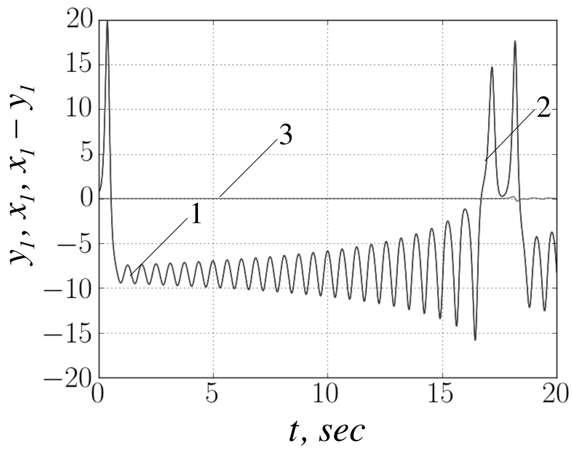

Fig. 1. Results of numerical sollution of (12) and (22) for x_1 variable.

The complete coincidence of the shown curves is clearly understood. This coincidence is approved by near zero values of error curve 3. Thus, we can claim the correct performing of transformation of the Lorenz equation into the canonical form by using the proposed approach.

The usage of the proposed approach ensures a coincidence of normal and canonical state spaces by only one variable. That is why other variables are differing. This difference cause different attractors in different state spaces. For example, a Lorenz attractor in the canonical state space and its projections are shown on Fig. 2. It is clearly understood the significant difference between the shown and well-known classical Lorenz attractors.

B. Numerical Transformation of The Lorenz Equations for x_1 Variable

We define the following functions as in (24).

$$\begin{aligned} F_1 &= \dot{y}_1 + \sigma y_1 - \sigma x_2; \\ F_2 &= \ddot{y}_1 - \sigma(\rho + \sigma)y_1 + \sigma(\sigma + 1)x_2 + \sigma y_1 x_3 \end{aligned}. \quad (24)$$

Let us transform (24) into discrete-time domain by using the simplest backward difference approximation of the differential operator:

$$\begin{aligned} \frac{d}{dt} &= \frac{1 - z^{-1}}{T}; \\ \frac{d^2}{dt^2} &= \frac{1 - 2z^{-1} + z^{-2}}{T^2} \end{aligned}, \quad (25)$$

where T is the sample time,

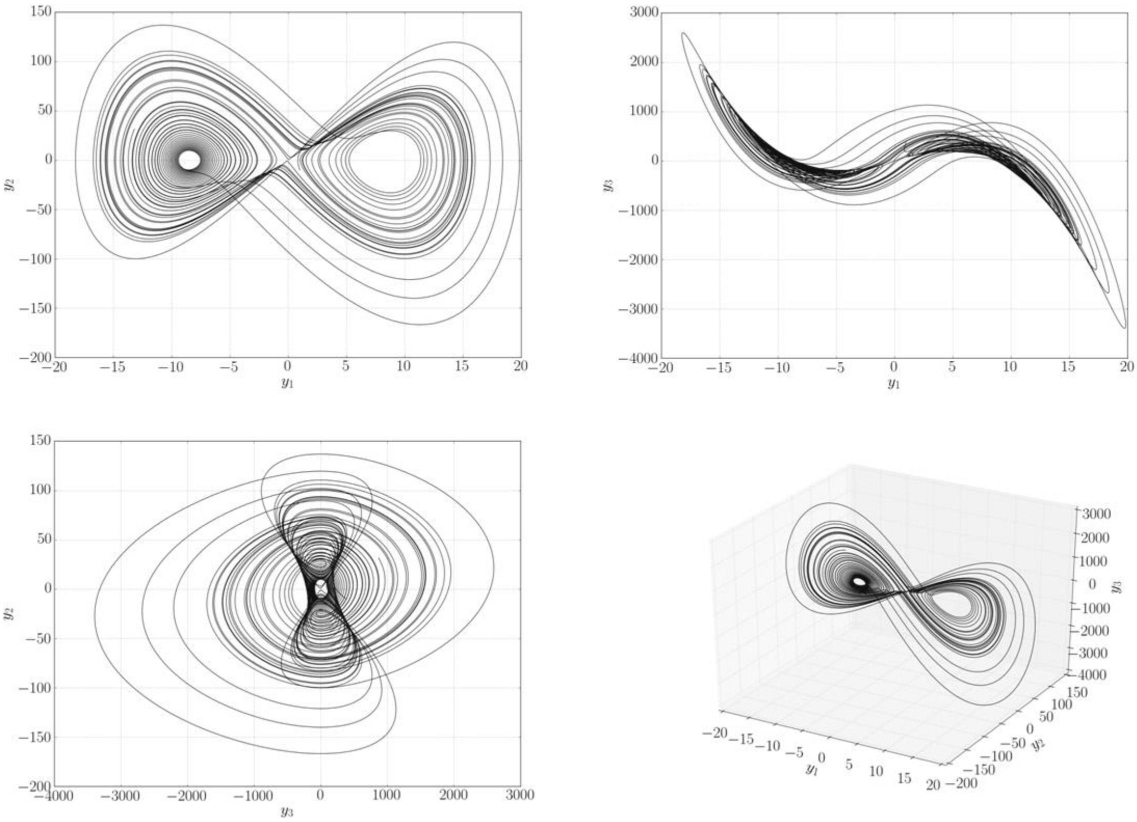

Fig. 2. Lorenz attractor in canonical state space.

as follows:

$$F_1 = \frac{y_1 - z^{-1} y_1}{T} + \sigma y_1 - \sigma x_2; \\ F_2 = \frac{y_1 - 2z^{-1} y_1 + z^{-2} y_1}{T^2} - \sigma(\rho + \sigma) y_1 + \sigma(\sigma + 1) x_2 + \sigma y_1 x_3 \quad (26)$$

or

$$F_1 = \left(\frac{1}{T} + \sigma\right) y_1 - \frac{1}{T} z^{-1} y_1 - \sigma x_2; \\ F_2 = \left(\frac{1}{T^2} - \sigma(\rho + \sigma)\right) y_1 - \frac{2}{T^2} z^{-1} y_1 + \frac{1}{T^2} z^{-2} y_1 + \sigma(\sigma + 1) x_2 + \sigma y_1 x_3. \quad (27)$$

At first, we define x_2 variable by using the following iterative algorithm based on Newton-Raphson method (28).

$$x_2 = z^{-1} x_2 + \frac{\left(\frac{1}{T} + \sigma\right) y_1 - \frac{z^{-1} y_1}{T} - \sigma x_2}{\sigma}. \quad (28)$$

This algorithm can be simplified as follows:

$$x_2 = \frac{z^{-1} x_2 + \left(\frac{1}{T} + \sigma\right) y_1 - \frac{z^{-1} y_1}{T}}{2}. \quad (29)$$

At last, we define x_3 variable by using similar procedure to (29) algorithm:

$$x_3 = \frac{z^{-1}x_3}{2} - \frac{\left(\frac{1}{T^2} - \sigma(\rho+\sigma)\right)y_1 - \frac{2}{T^2}z^{-1}y_1 + \frac{1}{T^2}z^{-2}y_1 + \sigma(\sigma+1)x_2}{2\sigma y_1}. \quad (30)$$

Equations (29)-(30) allows us to write down the following iterative canonical equations for the Lorenz system given in discrete-time domain:

$$\begin{aligned} y_1 &= z^{-1}y_1 + Ty_2; \\ y_2 &= z^{-1}y_2 + Ty_3; \\ y_3 &= z^{-1}y_3 - \sigma T y_1^2 x_2 + \sigma T\left((\beta + 2\sigma + 1)x_3 - 2\rho\sigma - \sigma^2 - \rho\right)y_1 + \sigma T\left(\rho\sigma + \sigma^2 + \sigma + 1 - \sigma x_3\right)x_2, \end{aligned} \quad (31)$$

where

$$\begin{aligned} x_2 &= \left(z^{-1}x_2 + (1/T + \sigma)y_1 - z^{-1}y_1/T\right)/2; \\ x_3 &= \frac{z^{-1}x_3}{2} - \frac{\left(1/T^2 - \sigma(\rho+\sigma)\right)y_1 - 2/T^2 z^{-1}y_1 + 1/T^2 z^{-2}y_1 + \sigma(\sigma+1)x_2}{2\sigma y_1}. \end{aligned} \quad (32)$$

Equations (31) and (32) are simpler than (22). These equations allow us to define both canonical y_i and normal x_i variables by solving the appropriate algebraic equations by using the following algorithm:

1. Current values of canonical variables y_1 and y_2 are defined by using the first and second expressions of (31).
2. Current values of normal variables x_2 and x_3 are defined by using (32) in iterative way.
3. Current value of canonical variable y_3 is defined by using the third equation of (31).
4. The cycle is repeated for all simulation time.

Similar to (23), we call equations (31)-(32) discrete-time Lorenz equations in canonical form.

It is clearly understood the simplicity of the proposed approach contrary to the solution of differential equations (22). Equations (31)-(32) depend on the sample time T as well as coefficients of equations (12). So, we claim the following statement:

Statement 2. The dynamic of the discrete-time Lorenz system in the canonical form depends not only on its parameters but also on the used numerical method.

This statement is proved by the numerical solution results of (31) and (32) for different sample time (Fig. 3-4).

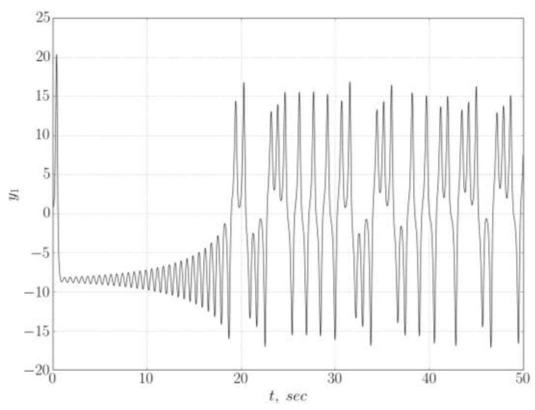

Fig.3 Results of the canonical Lorenz system simulation with sample time $T = 10^{-3}$ sec

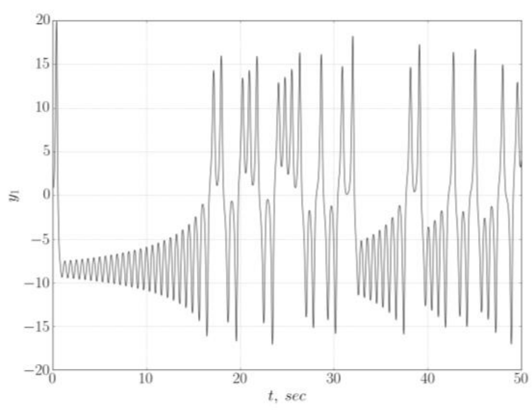

Fig.4 Results of the canonical Lorenz system simulation with sample time $T = 10^{-4}$ sec

We claim following as the result of all given mathematical expressions:

Vol. 3, No. 3, November 2017, pp. 117-124

Statement 3: If a dynamic system has a chaotic attractor in one state space, it has chaotic dynamic in other state spaces.

IV. Conclusion

The dynamic of a generalized chaotic system can be transformed into canonical form by defining n-th Lie derivatives and solving n-1 nonlinear algebraic equations. This transformation can be simplified by using numerical methods. One can develop numerical transformation algorithm as a part of controller software by using the mentioned numerical methods. The use of the proposed algorithm is one way of new chaotic attractors' discovering. These attractors can be obtained by transformation of known chaotic systems into various state spaces.

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